Limits of Stable Combustion in an Engine of Ultra-Small Spacecrafts

Kanysh O. Sabdenov*1, Johann Dueck1,2

*1L.N. Gumilyov Eurasian National University, Munaitpasov-Str. 5, 010008 Astana, Kazakhstan
2Friedrich-Alexander-University Erlangen-Nuremberg, Paul-Gordan-Str. 3, 91052 Erlangen, Germany
E-mail: sabdenobko@yandex.kz

Received: June 20, 2015; Sent for review: June 24, 2015; Accepted: November 18, 2015

Abstract

Using the analytical and numerical methods, an analysis of low-frequency stability of combustion in a micro-thruster rocket of solid propellant under the condition of a loss of heat from the combustion zone is conducted.

The problem of stability of the engine is solved by applying the phenomenological T0*-theory.

In a micro-thruster rocket, the domain of stable combustion becomes much narrower, due to occurrence of strong ω-in-stability, i.e. the exponential grow of perturbations with time. Nevertheless, there is a range of parameters, characterizing the laws of propellant combustion for which the sustained combustion may be possible.

Keywords

Rocket Micro-Engines on Solid Propellant, Efficient Initial Temperature, T0*-Theory, Unsteady Combustion, Instability of Burning

Nomenclature

cc – thermal capacity of solid propellant, J/(kg·K)
cp – gas thermal capacity by constant pressure, J/(kg·K)
E – effective activation energy of chemical reactions in the gas phase, J/mol
Ec – effective activation energy of the gasification reaction of the solid phase, J/mol
Fcr – minimum (critical) cross-section of the nozzle, m²
hw – wall thickness, m
j – heat flux, J/(m²·s)
k, r – phenomenological coefficients
L – heat of phase transition, J/kg
p – pressure, Pa
pe – pressure and its characteristic value in F. Williams’s equation, Pa
Re – internal radius of engine, m
R – gas constant, J/(kg·K)
Rg – universal gas constant, J/(K·mol)
S0 – surface area of combustion, m²
Tg – temperature of gasification surface of the solid propellant, K
T0 – initial temperature, K
T0* – effective initial temperature, K
Tc – temperature in the solid phase, K
Tc,av – “average” value of the propellant temperature, K
Tf – actual temperature of the flame (gas phase), K
Tb,ad – adiabatic temperature of flame (gas phase), K
Twall – temperature of the wall, K
Tch – temperature in the engine chamber, K
Td – characteristic temperature of dissociation, K
t – time, s
tg and tc – the characteristic times of the gas and solid phases of propellant, s
\[ u \] – burning rate, m/s
\[ V_k \] – volume of gas combustion products or combustion chamber volume, m³
\[ x(t) \] – coordinate of a surface of the gasification, m
\[ x(t) \] – coordinate of a thin zone of chemical reaction, m
\[ x \] – coordinate, m
\[ Y_s \] – concentration (volume fraction) of the product

**Greek Symbols**

\[ \alpha \] – heat transfer coefficient, W·m⁻²·K⁻¹
\[ \gamma \] – adiabatic index
\[ \kappa \] – thermal diffusivity of gas, m²/s,
\[ \kappa \] – thermal diffusivity of propellant, m²/s,
\[ \lambda_w \] – coefficient of heat conductivity of a wall, W·m⁻¹·K⁻¹
\[ \mu \] – molecular weight, kg/mol
\[ \mu \] and \[ \nu \] – phenomenological coefficients
\[ \rho \] – solid propellant density, kg/m³
\[ \lambda \] – heat conductivity coefficient of solid propellant, W·m⁻¹·K⁻¹
\[ \rho \] – gas density, kg/m³
\[ \sigma \] – Stefan-Boltzmann constant, W m⁻² K⁻⁴
\[ \tau \] – dimensionless time
\[ \omega \] – frequency of fluctuations, Hz
\[ \omega \] – real frequency (dimensionless)
\[ \Omega \] – complex frequency (dimensionless)

**Introduction**

With the appearance of ultra-small spacecraft [1, 2], the creation of micro-thrusters, running on solid propellant becomes important. They can be used for correction of the orbits, or for landing of spacecraft on asteroids and small planets. Besides, ultrafine motors can be used in microscopic military missiles.

Micro-thrusters are characterized by low ratio of volume to outer surface. At lower operating temperatures, this may lead to significant loss of heat from the propellant combustion zone. The relaxation time of gas-dynamic processes in the combustion chamber is also greatly reduced. These factors lead to new qualitative changes affecting the work of the micro-thruster.

Many relevant non-stationary processes in rocket engines: the establishment of a steady state burning, low-frequency and high-frequency sound fluctuations of the pressure and of the burning rate [3-10], or more complex vortex phenomena [11, 12], as a rule, are associated with the stability of steady state combustion of propellant in a combustion chamber of the rocket engine. Investigation of the stability is conveniently carried out on the basis of Zeldovich and Novozhilov theory (ZN-theory), where the basic laws are established from measurements of burning rate and temperature of surface of gasification of solid propellant in stationary mode of combustion [13, 14].

The phenomenological approach can also be formulated on the basis of the introduction of an explicit effective initial combustion temperature [15–17]. Here is given a summary of this approach.

From experiments on the observation of stationary burning functions

\[ u^0 = u^0(p, T_0), \quad T_0^0 = T_0^0(p, T_0), \quad u^0 = u^0(p, T_0^0) \]

are determined the dependences for burning rate \[ u^0 \] and temperature of gasification surface of the propellant \[ T_0^0 \] of initial temperature \[ T_0 \] and pressure \[ p \]. Steady state burning parameters are denoted by the superscript 0.

In unsteady combustion of solid propellant burning laws (1) can be written in the form

\[ u = u(p, T_0^0), \quad T_s = T_s(p, T_0^0), \quad u = u(p, T_0) \]

where \[ T_0^0 \] – is an effective initial temperature.
Let \( \omega \) be the characteristic frequency of fluctuations of thermodynamic or hydrodynamic parameters, with \( t_0 \) and \( t_c \) – the characteristic times of the gas and solid phases of propellant, so that their relation is small, \( t_0/t_c \ll 1 \). In the low-frequency region is \( \omega t_c \ll 1 \). This inequality allows us not to consider in detail the processes in the gas phase, which greatly simplifies the theoretical study.

For reasons of convenience, any pair of equations (1) and (2) can be used in equations (1) and (2). To carry out a theoretical analysis, it is often necessary to introduce the phenomenological coefficients \( k, r, \nu \) and \( \mu \):

\[
k = \frac{\partial \ln u^0}{\partial T_0}, \quad r = \frac{\partial T_s^0}{\partial T_0}, \quad \nu = \frac{\partial \ln u^0}{\partial \ln p^0}, \quad \mu = \frac{1}{T_s^0} \frac{\partial T_s^0}{\partial \ln p^0},
\]

where \( p^0 \) – is stationary base pressure.

The equation of heat transport is supplemented by the boundary condition for the heat flux at the gasification surface of the propellant (see below).

Previously, by other authors [18, 19] the temperature similar to \( T_0 \) was proposed to determine from the first equation of (2). In \( T_0 \)-theory [16, 17], this temperature is directly related to the instantaneous rate of burning and temperature gradient at the surface of the solid propellant, respectively, for an interval of the change of this temperature which is given an assessment.

**Physical Statement of the Problem**

Let us consider the problem of one-dimensional stability of the combustion in the rocket motor of cylindrical shape with a relatively large heat loss (Fig. 1). Solid propellant has a heat capacity \( c_v \), density \( \rho \), coefficient of heat conductivity \( \lambda \), and thermal diffusivity \( \kappa = \lambda/\rho c_v \).

![FIG. 1. SCHEMA OF A MISSILE MICRO-THRUSTERS WITH A CRITICAL SECTION \( F_c; R_c \) – INTERNAL RADIUS OF ENGINE; \( V_c \) – VOLUME OF GAS COMBUSTION PRODUCTS OR COMBUSTION CHAMBER VOLUME.](image)

The temperature of the propellant in the solid phase is denoted \( T_s(t, x) \).

On the surface at the coordinate \( x=0 \), occurs the gasification of solid phase at a temperature \( T_s \), the coordinate of a thin zone of chemical reaction is \( x=1 \). Temperature of the flame (gas phase) is marked as \( T_0 \).

In the most unfavourable conditions of outer space the propellant in a micro-thruster must burn at ultra-low initial temperature \( T_0 \approx 2.73 \text{ K} \). At the present time there are no experimental data indicating the possibility of burning at such low temperatures. We can only argue a reduction of combustion stability while reducing the initial temperature [13]. Therefore, below it is assumed that the condition for the existence of a stationary combustion [17] is fulfilled.

Heat flux \( j \) from the side surface of the propellant charge can be represented by the formula of Newton \( j = \alpha_s(T_s - T_0) \), where \( \alpha_s \) – is heat transfer coefficient.

The total heat transfer coefficient can be calculated on the basis of the rule of addition of thermal resistances [20]: the heat flow from the propellant itself, then the heat transport through the outer wall of the apparatus, and the heat (mainly by radiation) \( \alpha_s^{-1} = \alpha_c^{-1} + \alpha_{c1}^{-1} + \alpha_{r}^{-1} \). Here, \( \alpha_c^{-1} = 3\lambda_c/R_c \) results from the problem of cooling a cylinder.
(regular mode), $\alpha_{\omega}^{-1} = \lambda_{\omega}/h_{\omega}$ from the problem of heat transfer through the wall and

$$\alpha_{\text{eff}} = 4\sigma T_0^4 \left[ 1 + \frac{3}{2} \left( \frac{T_{c,\text{av}}}{T_0} - 1 \right) \right],$$

describes the heat loss due the radiation.

The loss of heat by radiation under the conditions of space dominates can be described by a nonlinear equation after the Stefan-Boltzmann: $j = \sigma(T_{\omega}^4 - T_0^4)$.

In this case it is difficult to obtain simple and descriptive analytical solution. The equation of heat loss can be linearized, representing $j = \alpha_{\text{eff}}(T_{\omega} - T_0)$ with an effective heat transfer coefficient $\alpha_{\text{eff}}$ with some “average” value of the propellant temperature $T_{c,\text{av}} = \text{const}$.

**Equation for the Energy in Solid Phase of the Propellant**

The combustion front moves with velocity $u = -dx/dt$. With the same speed, it moves the $x$-coordinate. In the laboratory coordinate system, combustion front can be described by the equation [17]:

$$\frac{\partial T_c}{\partial t} + u \frac{\partial T_c}{\partial x} = \kappa_c \frac{\partial^2 T_c}{\partial x^2} - \frac{2\alpha_S \delta}{c_c \rho_c R_c} (T_c - T_0).$$

Equation (3) allows the performing a complete analysis and gives a correct qualitative physical picture suitable for more complex types of heat loss.

Let the point $x = 0$ coincide with the gasification surface of solid propellant. Then the boundary and initial conditions are given by [17]:

$$x \to -\infty: \ T_c = T_0;$$

$$x = 0: \ T_c = T_s(t, p), \ \kappa_c \frac{\partial T_c}{\partial x} = g(T_0^4 - T_s^4),$$

$$g = \frac{1 + \sqrt{1 + 4\varphi}}{2}, \ \varphi = \frac{\kappa_c}{(u^0)^2} \frac{2\alpha_S \delta}{c_c \rho_c R_c};$$

$$T_c(0, x) = T_c^0(x).$$

Let us introduce the following dimensionless quantities:

- temperature $\theta = (T_c - T_0)/(T_0^\theta - T_0)$,
- burning rate $B = u/u^0$,
- surface temperature $\theta^s = (T_s - T_0)/(T_0^\theta - T_0)$,
- pressure $\eta = p/p^\theta$,
- time $\tau = t(u^0)^2/\kappa_c$ and coordinate $\xi = u^\theta x/\kappa_c$.

Equation (3) in these new variables takes the form

$$\frac{\partial \theta_c}{\partial \tau} + B \frac{\partial \theta_c}{\partial \xi} = \frac{\partial^2 \theta_c}{\partial \xi^2} - \phi \theta_c,$$

Equation (2) can now be brought in the form of: $B = B(\theta^s, \eta), \ \theta^s = \bar{\theta}^s(\theta, \eta)$ with the boundary and initial conditions

$$\xi = -\infty: \ \bar{\theta} = 0;$$

$$\xi = 0: \ \bar{\theta} = \bar{\theta}_s, \ \frac{1}{g} \frac{\partial \bar{\theta}}{\partial \xi} = B(\theta^s_0, \eta)(\bar{\theta}_s - \theta^s_0).$$

$$\bar{\theta}(0, \xi) = \bar{\theta}^0(\xi), \ \theta^0 = 1.$$
The stationary solution \( \theta^*(\zeta) \) of equation (4) must satisfy the following boundary conditions:

\[
\theta^*(- \infty) = 0, \quad \theta^*(0) = 1, \quad B = 1.
\]

A simple calculation yields

\[
\theta^*(\zeta) = \exp(g \cdot \zeta). \tag{6}
\]

Relative pressure fluctuation \( \delta \eta = (p - p^0)/p^0 \) are assumed to be very small. Therefore, it suffices to consider linear perturbations in temperature, combustion rate, etc., caused by changes in \( \delta \eta \). Their dependence on the time can be represented by an exponential function \( \delta \eta \sim e^{i \Omega t} \), with \( \Omega \) as a complex frequency.

The surface temperature is a function of the effective initial temperature and pressure \( \theta_c = \theta_c(\theta_c^*, \eta) \) and can be expanded in series:

\[
\theta_c \approx 1 + \frac{\partial \theta_c}{\partial \theta_c^*} \delta \theta_c^* + \frac{\partial \theta_c}{\partial \eta} \delta \eta.
\]

Since \( \delta \theta = \theta_c^* - 1 \), then using the definition of the coefficients

\[
r = \frac{\partial \theta_c}{\partial \theta_c^*} \approx \frac{\partial T_s^0}{\partial T_0}, \quad \mu = \frac{\partial \theta_c}{\partial \eta} \approx \frac{1}{T_s^0 - T_0} \frac{\partial T_s}{\partial \ln p},
\]

the equation for \( \delta \theta_c^* \) can be expressed as:

\[
\delta \theta_c^* = \frac{\partial \theta_c^*}{\partial \theta_c} \delta \theta_c + \frac{\partial \theta_c^*}{\partial \eta} \delta \eta = \frac{\delta \theta_c}{r} - \frac{\mu}{r} \delta \eta.
\]

Then, taking into account the resulting equation we obtain for the burning rate the following representation [10]:

\[
B(\theta_c^*, \eta) = 1 + \frac{\partial B}{\partial \theta_c^*} \delta \theta_c^* + \frac{\partial B}{\partial \eta} \delta \eta = 1 + \frac{\partial B}{\partial \theta_c^*} \left( \frac{\delta \theta_c}{r} - \frac{\mu}{r} \delta \eta \right) + \frac{\partial B}{\partial \delta \eta} \delta \eta.
\]

Here the derivations of the function of burning rate can be written using definition of the coefficients

\[
k = \frac{\partial B}{\partial \theta_c^*} \approx \frac{\partial \ln u^0}{\partial T_0}, \quad v = \frac{\partial B}{\partial \eta} \approx \frac{\partial \ln u^0}{\partial \ln p}
\]

as following:

\[
B = 1 + \frac{k}{r} \delta \theta_c + \left( v - \frac{\mu k}{r} \right) \delta \eta. \tag{7}
\]

The difference \( \theta_c - \theta_c^* \) included in the second boundary condition (5) must be also written in the linear approximation:

\[
\theta_c - \theta_c^* = 1 + \delta \theta_c - \delta \theta_c^* = 1 + (1 - 1/r) \delta \theta_c + \mu \cdot \delta \eta / r.
\]

Therefore, the second boundary condition in equation (5) after accounting the equality (7) after simple transformations takes the form

\[
\frac{1}{g} \frac{\partial \theta_c}{\partial \zeta} = 1 + \frac{k + r - 1}{r} \delta \theta_c + v \delta \eta. \tag{8}
\]

Let’s find the solution the differential equation (4).

Assuming that

\[
\theta_c = \exp(g \cdot \zeta) + \delta \theta_c, \quad \delta \theta_c = \delta \theta_c e^{i \Omega t}, \quad B = 1 + b e^{i \Omega t},
\]

where \( \delta \theta_c \) and \( b e^{i \Omega t} \) are small perturbations of temperature and burning rate with amplitudes \( \delta \theta_c(\zeta) \) and \( b \).

The use of equation (4), holding only linear values, leads to the equation
\[ \frac{d^2 \vartheta_c}{d \xi^2} - \frac{d \vartheta_c}{d \xi} - (\varrho + \Omega) \vartheta_c = b \frac{d \vartheta_c^0}{d \xi} . \]

Its solution corresponding to the condition \( \vartheta(-\infty) = 0 \) has the form

\[ \vartheta_c(\xi) = A \cdot e^{\xi} - b \frac{\varrho}{\Omega} \cdot e^{\xi} , \quad z = 1 + \frac{1}{2} (\varrho + \Omega) . \quad (9) \]

In equation (7) and in the boundary condition (8), let’s go over to the amplitude \( b \) of the perturbation of burning rate \( b \) and to the temperature of the propellant surface \( \vartheta(0) \):

\[ b \cdot e^{\Omega t} = \frac{k}{r} \vartheta_c(0) \cdot e^{\Omega t} + \left( \nu - \frac{\mu k}{r} \right) \delta \eta , \quad (10) \]

\[ \frac{e^{\Omega t}}{g} \frac{\partial \vartheta_c(0)}{\partial \xi} = \frac{k + r - 1}{r} \vartheta_c(0) \cdot e^{\Omega t} + \left( \frac{\mu}{r} + \nu r - \frac{\mu k}{r} \right) \delta \eta . \]

If there the formula (9) for \( \vartheta(\xi) \) is used, then after simple transformations two equations for the variables \( A \) and \( b \) can be obtained:

\[ A(z - ag) + \frac{k - 1}{r} \frac{g^2}{\Omega} b = (\mu + \Delta) \frac{g}{r} e^{-\Omega t} \delta \eta , \quad (11) \]

\[ A - \left( \frac{r}{k} + \frac{g}{\Omega} \right) b = -\frac{\Delta}{k} e^{-\Omega t} \delta \eta , \]

\[ a = \frac{k + r - 1}{r} , \quad \Delta = \nu r - \frac{\mu k}{r} . \]

The pressure fluctuation can be presented similar as \( (\eta_0 \text{- amplitude}) \)

\[ \delta \eta = \eta_0 \exp(\Omega t) . \quad (12) \]

From the equations (11) the relationship between the amplitudes of the burning rate \( b \) and pressure \( \eta_0 \) can be found as:

\[ b \left[ \frac{r}{k} + \frac{kg}{\Omega} \right] (z - g) - g(k - 1) = [\nu g + \Delta (z - g)] \eta_0 . \quad (13) \]

The Equation for the Pressure in the Combustion Chamber

Modeling of processes in the gas phase and the mechanism of the decomposition the solid phase of a propellant represents a separate complex and important task. This was the subject in a large number of studies [21-27 et al.]. Even modeling the properties of homogeneous propellant [21, 22] is fraught with difficulties.

Therefore, it is of interest when the most general laws of combustion are used. The simplest models are given by Belyaev-Zel’dovich [13, 14], Merzhanov-Dubovitskii [27], and Denison-Baum [21]. They can be interpreted as special cases of the model of Williams [22]:

\[ u = \text{const} \cdot \exp \left( - \frac{E_c}{2RT_s} \left[ 1 - \frac{pY_s}{p_e \exp \left( \frac{\mu L}{RT_s} \right)} \right] \right) . \]

Here are: \( E_c \) – effective activation energy of the gasification reaction (pyrolysis) of solid phase of the propellant; \( R \) – universal gas constant; \( p \) and \( p_e \) – pressure and the characteristic value of \( p \) when the heat of phase transition of evaporation \( L \) dominates the pyrolysis; \( Y_s \) – concentration (volume fraction) of the product of gasification with a molecular weight \( \mu \).

According to Williams [22], at low pressures \( p \ll p_e \) the gasification of propellant occurs like pyrolysis (model of Merzhanov-Dubovitskii [27], and Denison-Baum [21]) and the burning rate is \( u = \text{const} \cdot \exp(- E_c/2RT_s) \).
At high pressures $p \gg p_c$, the gasification propellant is like the evaporation (model of Belyaev-Zeldovich) and the concentration $Y_s$ and the temperature $T_s$ on the surface are linked by Clausius-Clapeyron relation: $Y_s = p_p \exp(-\mu L/RT_s)$.

For a relatively slow time-dependent processes ($\omega t_b << 1$), the results of investigating the stability of combustion produced using by the models Belyaev-Zel’dovich and Merzhanov-Dubovitskii or Denison-Baum, [6–8, 10] are almost equivalent with the accuracy of the order $\omega^2 t_b^2$.

Thus, when the inequality $\omega t_b << 1$ is valid then the gasification mechanism of propellant and processes in the gas phase are not significant till small quantities of the order $\omega^2 t_b^2$. Therefore, the study of stability in the range of frequencies $\omega \ll 1/t_b$ can be done using the simplest models of the gas phase in the combustion chamber of the engine.

In the simplest model of the gas phase in the combustion chamber, only the change of pressure and density [13, 14, 28] is taken into account.

Let $S_p$ – be the surface area of combustion. Typically, the characteristic time (maximum) of unsteady processes is such that the burning surface during this period varies slightly. Therefore we can assume that the $S_p \approx \text{const}$. The mass flow velocity of burned propellant is $\rho u S_p$.

The mass of gas, escaping via the engine nozzle per unit of time is equal to $A_c p F_{cr}$, where $A_c = \text{const}$, and $F_{cr}$ is the minimum (critical) cross-section of the nozzle.

The rate of change of gas mass $\rho V_k$ in the chamber with a volume $V_k$ is determined by the difference between the gas which comes from the combustion and the amount of substances exiting through the nozzle:

$$
\frac{d(\rho V_k)}{dt} = \rho_c u S_p - A_c p F_{cr},
$$

$$
A_c = \frac{\Gamma(\gamma)}{\sqrt{R_g T_k}}, \quad \Gamma(\gamma) = \left[\frac{2}{\gamma + 1}\right]^{\frac{\gamma + 1}{\gamma - 1}},
$$

where are: $R_g$ – gas constant; $\gamma$ – adiabatic index; $T_k$ – temperature in the engine chamber.

In absence of solid particles (metals, their oxides, etc.), the density of the gas can be determined from the equation of state of ideal gas $\rho = p/RT_k$.

Under unsteady conditions in the combustion chamber, the changes in the density and pressure of the gas are significantly superior to the change in temperature, so that $T_k \approx \text{const}$ can be taken [28].

Then equation (14) can be written as

$$
\frac{V_k}{R_g T_k} \frac{dp}{dt} = \rho_c u S_p - A_c p F_{cr}.
$$

The stationary pressure $p^0$ in the combustion chamber according to burning rate $u = u_0(T_0 - p/p_0)^{\nu}$, ($p_0$ – is the initial pressure) is determined from the condition $dp/dt = 0$:

$$
p^0 = \frac{\rho_c u_0 S_p}{A_c F_{cr}} = p_0 \left(\frac{\rho_c u_0 S_p}{A_c F_{cr}}\right)^{1-\nu}.
$$

In dimensionless form, equation (15) contains only one the devise constant $\chi$ [13, 14, 28]:

$$
\chi \frac{d\eta}{dt} = B - \eta,
$$

$$
\chi = \frac{(u_0^0)^2 V_k}{\kappa_c R_g T_k A_c F_{cr}}.
$$
with the ratio $t_s/t_c \approx t_s/t_v$ with

$$t_s = \frac{1}{V_k} R_s T_s A_s F_s, \quad t_c = \frac{\kappa_c}{(u_0^w)_v^2}.$$ 

**Dispersion Equation**

From equation (16) for non-stationary component of the pressure follows:

$$ \chi \frac{d\delta \eta}{d\tau} = b \cdot e^{\Omega \tau} - \delta \eta. $$

Taking into account the representation of the pressure by equation (12), the following algebraic equation can be obtained [13, 14]:

$$ b = (1 + \chi \Omega) \eta_a. $$

(17)

For a finding of sustainable combustion, the system of equations (13) and (17) with unknown $b$ and $\eta_a$ must have a nontrivial solution. This is possible if its determinant is zero. The vanishing of the determinant gives the dispersion equation for the frequency $\Omega$:

$$ \left[ (z - g) \left( r + \frac{kg}{\Omega} \right) - g(k - 1) \right] (1 + \chi \Omega) - g \nu - (z - g) \Delta = 0. $$

(18)

If $\varphi = 0$, then $g = 1$ and $z = (1 + (1 + 4\Omega)^{1/2})/2$. Then from equation (18) follows equation obtained by Novozhilov [13, 14].

Assume the dependence of the burning rate only on the surface temperature: $u(T_s)$. This means that in the experiments found dependences of burning rate on pressure, for example, the form $u = u_b(T_b)(p/p_0)^{\nu}$, results from two factors [13, 14]:

1) the existence of a relationship $T_s(p, T_0);

2) the vanishing of the Jacobian, i.e.

$$ J = \frac{\partial (u, T_s)}{\partial (p, T_0)} = \frac{\partial u}{\partial p} \frac{\partial T_s}{\partial T_0} - \frac{\partial u}{\partial T_s} \frac{\partial T_s}{\partial T_0} = 0. $$

The condition that the Jacobian is equal to zero in dimensionless variables can be written as

$$ J' = \frac{\partial (B, \theta_s)}{\partial (\eta, \theta_0)} = \frac{\partial B}{\partial \eta} \frac{\partial \theta_s}{\partial \theta_0} - \frac{\partial B}{\partial \theta_s} \frac{\partial \theta_s}{\partial \theta_0} = \Delta = v + \mu k = 0. $$

Note, in practice usual is valid that $v \sim k \sim 1; \quad r \sim \mu \ll 1$. At the present level of experimental techniques can be found that $v' \sim 10^{-1} \ldots 10^{-2}$. Such values are comparable to experimental errors [13] and accordingly allows to take equality $\Delta = 0$.

Therefore, equation (18) can be written in the form

$$ \left[ \left( \frac{z - 1}{g} \right) \left( r + \frac{kg}{\Omega} \right) - k + 1 \right] (1 + \chi \Omega) - v = 0. $$

(19)

The combustion chamber volume in micro-thruster $V_c \rightarrow 0$ and critical section $F_c \rightarrow 0$, but so that the ratio $V_c/F_c \rightarrow 0$. Therefore, for such engine parameter is valid that $\chi \ll 1$. Investigation of stability at low values of $\chi$ by analytical and numerical methods, but without the loss of heat from the solid propellant ($\varphi = 0$) is presented in [29-31]. From the results of [29, 30], it implies the possibility of stable operation of the micro-motor both when $v < 1$ and $v > 1$.

Equation (19) relative to the variable $\Omega$ can have multiple roots. Combustion mode is unstable if at least one of the roots leads to unstable behavior.
If \( r \neq 0 \), then oscillatory properties of burning propellant may occur [13, 14]. On the curve of parameters for which such properties will be observed, the real part of the frequency \( \Omega \) is equal to zero and \( \Omega = i\omega \), where \( i = (-1)^{1/2} \) – complex imaginary unit and \( \omega \) – real frequency. The perturbations on this curve (neutral) do not grow or decay. Often, such a curve is the boundary of neutral stability.

For building the curve of neutrality demands to express the coefficients \( k, \nu \) as parametric functions of frequency \( \Omega \):

\[
\begin{align*}
    k &= \frac{\text{Im}\left(\frac{v}{1 + \chi \Omega} \frac{1}{z/g - 1}\right)}{\text{Im}\left(\frac{z/g - 1}{z/g - 1}\right)}, \\
    r &= \frac{\text{Im}\left(\frac{v}{1 + \chi \Omega} \frac{1}{(z/g - 1)/\Omega - 1}\right)}{\text{Im}\left(\frac{z/g - 1}{z/g - 1}/\Omega - 1\right)}, \\
    \nu &= \frac{\text{Im}\left[\frac{r(z/g - 1)}{(z/g)/\Omega - 1}\right] + \text{Im}\left[\frac{1}{(z/g)/\Omega - 1}\right]}{\text{Im}\left(\frac{1 + \chi \Omega}{z/g - 1}/\Omega - 1\right)}.
\end{align*}
\]

Here operation \( \text{Im} \) means taking the imaginary part. One can eliminate here the variable \( \Omega \), taking any pair of equations and thus build the necessary dependence, for example, \( r(k) \) or \( \nu(k) \).

The values \( r, k, \nu \) calculated according to the given formulas may occur to be negative. The parameters \( r, k \) may be less than zero in two cases [15, 16]:

1) the gasification reaction of solid phase is strongly endothermic. Then the reaction heat provides surface temperature \( T_s \) lower than the initial temperature \( T_0 \);

2) the excitation of oscillatory energy and dissociation in the gas flame are so strong that the effective activation energy \( E \) varies considerably with increasing of temperature. Under quadratic law \( E = E_0 (1 + T/T_s) \) proposed in [32], where are: \( E_0 \) – conditional activation energy at zero temperature and \( T_s \) – characteristic temperature of dissociation, the theoretical values of the phenomenological coefficients \( r \) and \( k \) can be calculated by the following formulas [15, 16]:

\[
r = \frac{E}{E_0} \sigma, \quad k = \frac{E(T_s - T_0)}{2RT_s^2} \sigma, \quad \sigma = \left(\frac{T_s}{T_b}\right)^2 - \left(\frac{T_b}{T_s}\right)^2.
\]  

(20)

For \( \sigma > 0 \) the parameter \( k < 0 \), if \( T_s < T_b \) (the first case). The validity of inequality \( T_s < T_b \) leads to negative \( r \) and \( k \), if \( T_s > T_b \) (the second case). Both cases are exotic for physics of combustion, but it is interesting from theoretical point of view. It was shown by the analysis using a detailed simulation of combustion [15, 16], that for such cases exist sustainable solutions.

Below we restrict ourselves to the values \( r > 0, k > 0, \nu > 0 \), because there are currently no experimental data indicating the presence of phenomenological coefficients with a negative sign.

State of combustion is determined by three parameters \( k, r \) or \( \nu \). Graphically it can be displayed in a space of three coordinate axes, each axis belongs to one of the variables \( k, r \) or \( \nu \). If we consider any pair of these variables, the stability boundary is a line, as shown in Fig. 2. If we consider all three variables, the boundary appears as a surface.

We emphasize here the following. Stationary (stable) burning propellant assumes the suppression of any fluctuations in temperature or pressure in a system “combustion chamber – propellant”. Linear analysis of fluctuations of small amplitude applied in the article may indicate the limit set of parameters on which this condition breaks.

In the space of parameters describing the system, the configuration of stability boundary is complex. But even building it in a multi-dimensional space of parameters, we can not predict with certainty the behaviour of the system at some distance from the boundary. Within the stability region, combustion is stable, but outside of it, this may be extinction, uncontrolled flare-up or burning with finite amplitude fluctuations.

This can be detected during the application of non-linear analysis. It should be noted that apart from the boundary of stability due to feedback between the fluctuations and stable state of the system, itself stable state of the system may be disturbed by the mechanism of the thermal imbalance in the combustion zone.
**Λ-Instability and \( a_0 \)-Instability**

Let’s perform the approximate calculation of the unsustainable combustion.

In equation (19) \( \chi = 0 \) is set. Then the new equation

\[
\left( \frac{z}{g} - 1 \right) \left( \frac{g k}{\Omega} \right) - k + 1 - v = 0
\]

after simple transformations reduces to the form:

\[
\Omega^2 + \frac{g^2 \Lambda}{r^2} \Omega + \omega_0^2 = 0,
\]

(21)

Here two parameters:

\[
\Lambda = \frac{r}{g} \left[ 2k - (2g - 1)(v + k - 1) \right] - (v + k - 1)^2, \quad \omega_0^2 = \frac{kg^2 [k - (2g - 1)(v + k - 1)]}{r^2}
\]

are introduced.

Such dispersion equations of the second degree indicate the presence of second order non-linear differential equation for the non-stationary temperature \( \theta(t) \) of gasification surface and the burning rate \( B(t) \) [15, 16, 32, 33].

From here there is the possibility of analysis of unsteady combustion modes, using interpretation of the theory of dynamical systems [34]. Thus, in equation (21) the complex \( g^2 \Lambda/r^2 \) can be interpreted as the “viscosity”, \( \omega^2 \) – as a returning force.

According to theory of dynamical systems, if \( \omega^2 > 0 \), then combustion is unstable when the inequality \( \Lambda < 0 \) is valid (i.e., dynamical systems [34] with “negative viscosity”, hence the name “Λ-instability”). This inequality can be written as

\[
\frac{g(v + k - 1)^2}{2k - (2g - 1)(v + k - 1)}.
\]

The boundary of instability is found from the equation \( \Lambda = 0 \), it is given by the quadratic equation with respect to the complex \( v + k - 1 \):

\[
g(v + k - 1)^2 + r(2g - 1)(v + k - 1) - 2rk = 0.
\]

The solution of this equation gives the boundary of instability. Two roots of this equation are given by the expressions:

\[
v_{1,2} = 1 - k + \frac{-r(2g - 1) \pm \sqrt{r^2(2g - 1)^2 + 8rg}}{2g}.
\]

(22)

The root \( v_1 \) corresponds to the negative sign before the radical, the root \( v_2 \) – to the positive sign.

The values from the interval \( v_1 < v < v_2 \) correspond to stability condition.

But the dispersion equation (21) is obtained as a result of transformations to get rid of radicals, so the equation (21) may give “parasitic” solutions. Such a solution is the root \( v_1 \).

Let us consider a special case.

The boundary of instability as a function \( v(k) \), which is built with the values of the parameters \( \varphi = 1 \ (g = 1.618), \ r = 0.1 \) and \( r = 0.15 \) is shown in Fig. 2.

The upper and the lower parts of the curves in Fig. 2 are built by the formulas (22).

The “parasitic” solution is shown by lower dashed curves FD in Fig. 2.
To construct a complete picture of the considered physical phenomenon, we take into account that the instability may occur as a result of the acquisition of a negative sign of the natural frequency \( \omega^2 \) (so called \( \omega^2 \)-instability).

Therefore, to complete the analysis, the region bounded by the two curves \( v_1(k, g) \) and \( v_2(k, g) \), must be supplemented by another condition of stability loss \( (\omega^2 < 0, \text{for considered case } \varphi = 1) \),

\[
v > 1 - k \frac{2(g-1)}{2g-1} = 1 - 0.55k.
\]

In Fig. 2, this inequality corresponds to the area above the straight line \( AG \). As a result, the region of stable operation of the micro-motor at low frequencies covers an area which lies partly below the line \( AB \) and partly below the curve \( BC \). Moreover, the loss of stability above the curve \( BC \) happens with the oscillations in the rate of combustion, but above the straight line \( AB \) oscillations are absent.

It is possible to come to these results in a different way: let the frequency \( \Omega \) to be a real number. Near the stability limit is valid \( \Omega \ll 1 \). Using in equation (9) Taylor series expansion of the parameter \( z \) in powers of \( \Omega \):

\[
z \approx g + \frac{\Omega}{\sqrt{1 + 4\varphi}} - \frac{\Omega^2}{(1 + 4\varphi)^{3/2}} = g + \frac{\Omega}{2g-1} - \frac{\Omega^2}{(2g-1)^3},
\]

from equation (19) follows:

\[
v = 1 - k \frac{2(g-1)}{2g-1}. \tag{23}
\]

It coincides with the previous equation, when in it the sign of inequality is changed to equality.

Thus, the only boundary of instability without oscillations corresponds to equality \( \omega^2 = 0 \). This conclusion is confirmed by the numerical solution of the equation (19), with \( \chi \neq 0 \).

Area of \( \Lambda \)-instability goes to the limit value if reducing the heat loss from the combustion zone, but does not disappear, while ensuring perfect thermal insulation of the combustion zone \( (\varphi = 0, g = 1) \).

Let’s consider some results of numerical analysis of the equation (19) of determining the conditions of \( \Lambda \)-instability (the origin of this term see below) taking the frequency \( \Omega = \nu_0 \).

For illustration we take the following values \( \chi = 0.01; \ r = 0.2; \ \varphi = 1 \). The results are presented in Fig. 3.

In the range of frequency \( \nu \) from 0 to \( \nu_{\text{max}} = 6.3 \), there is a neutral curve \( BC \). Segment \( AB \) is a part of a straight line \( AG \) with the equation \( \nu = 1 - 0.55k \), (special case of equation (23)). For frequencies \( \nu > \nu_{\text{max}} \) is the parameter \( \nu < 0 \).

While with the increase of the heat loss from the combustion zone (if \( \varphi \) increases), the area of occurrence of \( \Lambda \)-instability reduces but the area of \( \omega^2 \)-instability increases (Fig. 4). But general the region of stable burning is reduced.

In the range of \( 0 < k < 0.4 \) (curve 1 in Fig. 4), there are no oscillatory solutions.
Part 2 of the curve 2 in Fig. 4 (0 < k < 0.77), which is the boundary of $\omega^2$-instability, is built by the formula (22) $v = 1 - 0.72k$.

When comparing the curves in Fig. 2 and Fig. 4, build with $r = 0.1$, their notable difference can not be detected. Here already $\chi\omega_{\text{max}} \approx 0.14$ can not be considered much less than unity.

Increase $\varphi$ from 1 to 3 leads to an increase $\omega_{\text{max}}$ from 14.2 to 15.2. It is also found that with good accuracy $\omega_{\text{max}} \sim 1/r$.

Increase of heat loss narrows the field of sustainability, but only up to certain limits. For each value of $r$, there exists such $\varphi = \varphi_c$ above which the boundary of the stability region is set to line $v = 1 - \text{const}\cdot k$. For $\varphi > \varphi_c$, the imaginary part of the root $\Omega$ is equal to zero and there is no oscillatory instability. For example, when $r = 0.1$ and 0.2, respectively $\varphi_c \approx 4$ and 12.

Comparison of the results obtained by the formula (22) for $v = v_2$ and numerical solution of the equation (19) shows the good accuracy of this analytical expression, except for a small vicinity near $v = 0$.

**$\omega_\phi$-Instability and Condition for the Existence of Combustion**

This kind of instability in contrast to the above discussed $\Lambda$-instability develops with increasing heat losses from the combustion zone in the gas phase.

From the dispersion equation for $\chi = 0$, it follows that if $\omega^2 < 0$, then combustion is always unstable (i.e. for any sign of $\Lambda$) and oscillatory solutions for perturbations are absent.

The requirement for the stability $\omega^2 > 0$, or $(2\varphi - 1) = (1 + 4\varphi)^{1/2} \leq k/(v + k - 1)$, leads to the existence of critical value of $\varphi_c$. His excess will cause $\omega$-instability, and that such no instability appears it is necessary

$$\varphi < \varphi_c = \frac{1}{4} \left[ \frac{k^2}{(v + k - 1)^2} - 1 \right].$$

(24)
In practice, it is required to create a stable engine running on propellant with desired properties. Therefore, the condition (24) is better to express in the form of restrictions on the heat transfer coefficient \( \alpha \) and the radius of engine \( R_c \):

\[
\alpha < \frac{c_p(m^0)^2}{8 \rho \kappa} \left[ \frac{k^2}{(v + k - 1)^2} - 1 \right] \quad m^0 = \rho_c u^0.
\] (25)

Condition given in (25) to determine the stability region, makes sense if the other condition of existence of stationary combustion of gas in narrow tubes [35] is met.

Assume the chemical reaction with a single effective activation energy \( E \). Gas thermal capacity possess a constant pressure \( c_p \), density \( \rho \) and coefficient of thermal diffusivity \( \kappa \). The flame in gas has the adiabatic temperature \( T_{b, ad} \).

Heat losses result in lower combustion temperatures below this, adiabatic. Since the burning rate is highly dependent on the combustion temperature, its reduction leads, as shown by the analysis [35] to the breakdown of combustion. Using the results of [35], the condition for the existence of stationary combustion in the combustion chamber is:

\[
\alpha < \frac{1}{4e} \frac{c_p(m^0)^2}{\rho \kappa} \frac{RT_{b, ad}^2}{E(T_{b, ad} - T_s)} \quad e \approx 2.72.
\] (26)

Assuming \( T_d >> T_{b, ad} \), from (20) we obtain [16, 36]:

\[
k = \frac{E(T_s - T_0)}{2RT_{b, ad}^2}.
\]

Now, the inequality (26) takes the form

\[
\alpha < \frac{1}{8e} \frac{c_p(m^0)^2}{\rho \kappa} \frac{T_s - T_0}{k(T_{b, ad} - T_s)}
\] (27)

or in the form of a limit on the value of the parameter \( k \):

\[
k < \frac{1}{8e} \frac{c_p(m^0)^2}{\rho \kappa} \frac{R_c}{T_{b, ad}} \frac{T_s - T_0}{T_{b, ad} - T_s}
\] (28)

According to inequality (28), the region of sustainable work of the micro-thruster in Fig. 4 can be further split by vertical lines. As a result, the stability region narrowed even more.

Comparing the result (27) with the condition (25), we see that \( \omega \)-instability can occur if

\[
\frac{1}{e} \frac{c_p \rho_c \kappa}{c_p \rho \kappa} \frac{T_s - T_0}{T_{b, ad} - T_s} > k \left[ \frac{k^2}{(v + k - 1)^2} - 1 \right].
\]

In the rocket combustion chamber \((T_s - T_0)/(T_{b, ad} - T_s) \sim 0.1; c_p / \rho_c \sim 1; \rho \kappa / \rho \sim 10^3; \kappa / \kappa \sim 0.01\). Therefore, as in the left-hand side, as in the right part, there is a number at order of unity. Therefore, the above inequality can be satisfied if \( v + k \approx 1 \).

**Conclusion**

In a rocket micro-thrusters when the parameter \( v < 1 \), different kinds of manifestations of unstable combustion can be met: A-instability with a characteristic amplitude decay of burning rate and \( \omega \)-instability, which is characterised by sustained oscillations of burning rate. The first one is weakened by increasing of the parameters \( r \) and \( \phi \), the second one – amplified.

Their presence, particularly \( \omega \)-instability poses a number of technical difficulties in creating of micro-thrusters.
In the case of $\Lambda$-instability, the growth of perturbations occurs in an oscillatory manner, combustion can reach the self-oscillating mode.

Extinction might not occur and the engine continues to run.

For a $\omega$-instability, the growth of disturbances occur without fluctuations, and at high heat loss from the combustion zone, it will lead to extinction.

New equations for the boundaries of the region of stable combustion in the combustion chambers of ultra-small rocket engines are derived.

The change in the sign of the parameter $\varphi$ can be formally interpreted as a case of having preheating of the combustion zone. When heating, the combustion in the engine can become more sustainable.

Such an interpretation of the parameter $\varphi$ is possible (and useful in the sense of generality of formulation of the problem), although it cannot fully reflect the heating of propellant from the outside.

One option could be a heat flow from a gas flame zone along the walls charge-containing cells. A prerequisite for this is to have a high thermal diffusivity of the wall material, much larger than the thermal diffusivity of the propellant itself.

Another variant of amplification the propellant heating may be use of conductive elements (e.g. metal wires) extending through the propellant toward the combustion front. Contacting one end with a gas flame, such elements also contribute to the rapid transfer of heat deep into the fuel.

For the design of ultra-small micro-thrusters for spacecraft of interest are the propellants, in which the parameters $v$ and $k$ are much smaller than unity.

The reason for this is that there are restrictions on the parameter $k$ at low initial temperatures, as well as the achievement the limiting border $v + k < 1$ of the instability region with an increase in heat loss from the combustion zone.

From the limit of stability conditions $v + k < 1$ follow the restrictions $k < 1$ и $v < 1$.

These inequalities will be more correct, considering the decrease in the specific heat of solids ($c_s$) with a very low initial temperature $T_0$ (Debye-Einstein equation).

The presented method of the analysis and the results can be applied to the study of stable combustion in hybrid rocket micro-motor [37-39]. In this case, the main feature of the hybrid combustion engines – the strong dependence of the combustion rate of the solid propellant of the blowing by gases, is not an important factor. Gasification of solid propellant is determined by universally temperature $T_0$, its quantity primarily depends on the ratio between the flow of heat from the gas phase to the surface of the gasification and the heat flux in the solid phase.

**REFERENCE**


