Analytical Heat Transfer Analysis under Boundary Conditions of the Fourth Kind (Conjugate)

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Abstract
The conjugate heat transfer approach is a contemporary powerful tool for solution of heat transfer problems, and numerous results obtained numerically for different particular industrial and natural topics are now available. In contrast, this article presents analytical solutions of laminar (exact) and turbulent thermal boundary layer equations and formulate general features of conjugate convective heat transfer. These analytical results differ in principle from those gained by common methods based on heat transfer coefficients which in fact are empirical means. Two forms, in series and in integral, of analytical expressions for heat flux at arbitrary body surface temperature provide the high accurate calculations using the first form when the series converges fast applying the second one otherwise. On the base of these fundamental relations, the simple methods of solution of the conjugate heat transfer problems for thin and thermally thin plate are developed. Analysis of presented solutions shows that: (i) the surface temperature head variation and Biot number for isothermal surface are the basic characteristics defining the conjugate heat transfer intensity, (ii) the effect of temperature head gradient on the heat transfer coefficient is similar to influence of velocity gradient on friction coefficient: the positive gradients in both cases lead to growing coefficients, and decreasing velocity or temperature head results in lessening of appropriate coefficients and separation flow or heat flux inversion which is analogous to separation phenomenon, (iii) the basic relation in the form of series is a general boundary condition containing known particular cases.

Keywords
Analytical Solutions, Laminar Flows, Turbulent Flows, Conjugate Heat Transfer, Gradient Analogy

Introduction
It has been long known that exact solutions of convective heat transfer problems require satisfaction of the boundary conditions of the forth kind which are in essence the same as conjugate conditions. Such boundary conditions consist of the conjugation of the body and fluid temperature fields at their interface. However, in practice before computer advent, availability of problem solutions with such boundary conditions was restricted only to very simple cases. Due to that instead of exact formulation, a boundary condition of the third kind has been used since the time of Newton. In fact, this means that exact relations of the fourth kind are substituted by assumption that a heat flux \( q_w \) through a solid-fluid interface is proportional to difference between the body temperature \( T_w \) and the fluid temperature \( T_\infty \) far away from a solid

\[
q_w = h (T_w - T_\infty)
\]  

In such approach, the accuracy rests on a successful determining a proportionality factor-the heat transfer coefficient \( h \). Since there was no well-grounded theoretical heat transfer coefficient estimation, until the last few decades only, experimental results might be used as a reliable data. Therefore, in practical calculations, the heat transfer coefficient was usually estimated from available experimental data or otherwise the values for the simplest cases of constant interface heat flux or temperature were employed. Such simplified approach, when the effect of actual wall temperature distribution on the interface is neglected, was acceptable before the computers came to use resulting in significant increasing of the computing accuracy.
Practically at the same time, at the middle of 1960s, the interest in the conjugate convective heat transfer was
developed, and since that time, many convective heat transfer problems have been considered using a conjugate,
coupled, or adjoint statement. These three equivalent terms correspond to situation when the solution domain
consists of two or more subdomains in which parts of studied phenomena are described by different differential
equations. After solving the problem in each of subdomains, these solutions should be conjugated. The heat
transfer between a body and a fluid flowing past it is a typical conjugate problem, because the heat transfer inside
the body is governed by the elliptic Laplace equation or by the parabolic differential equation, while the heat
transfer inside the flowing fluid is governed by the elliptic Navier-Stokes equation or by the parabolic boundary
layer equation. The solution of such conjugate problem gives the temperature and heat flux distributions along the
body-fluid interface, and there is no need for a heat transfer coefficient. Moreover, heat transfer coefficient can be
computed using these data.

The conjugate procedure is needed also if the problem is governed by only one differential equation, but the
subdomains problems contain different materials or other not identical properties. For example, the transient heat
transfer between a hot plate and a cooling thin fluid film flowing along it is a conjugate problem. In such a case, the
plate at each moment is divided into a wet part covered with moving film and a dry one which is still uncovered.
Although heat transfer in both portions of the plate is governed by the conduction equation, the solutions for each
of these parts differ from each other and should be conjugated, because thermal properties of wet and dry portions
are different.

Numerous publications show that conjugate methods in heat transfer starting in the sixties of the last century with
simple problems, like [1-3], now became a powerful tool for modeling and investigating problems of wide scope
from rockets, nuclear reactors, turbomachines and other engineering and industrial systems [4-13] to different
technology processes such as drying, thermal material treatment, food production [13-21], and biology and medical
processes [22, 23]. More than 200 solutions of conjugate heat transfer problems from early publications to modern
results are reviewed in the author’s book [24] showing applicability of this modern approach.

There are many conjugate problems not only in heat transfer, but also in other areas of science and technology. For
instance, studying subsonic-supersonic flows requires conjugation, because the subsonic flow is governed by
elliptic or parabolic differential equations, while the supersonic flow is described by hyperbolic differential
equation. Combustion theory and biological processes are the other two examples. Every combustion process has
two areas containing fresh and burnt gases with different properties. An example of treating combustion as
conjugate problem is given in [25]. In biology, the diffusion processes usually proceed simultaneously in
qualitatively different areas (through membranes), and, therefore, require a conjugation procedure. The flow of
blood or urine in human organs is another example of typical conjugate problem in biology, because such flow
exists due to peristaltic motion which occurs as a result of interaction between the elastic vessel walls and fluid
inside it. Similar studies in mathematics, called mixed problems, began to be considered much earlier than
conjugate physical systems. The one of the first parabolic/ hyperbolic mixed equation was studied by Tricomi in
1923.

It follows from available literature that practically any heat transfer problem may be solved by contemporary
numerical conjugate methods if a relevant mathematical model could be formulated. Each of such solution gives a
particular reliable fundamental or applicable result for some specific problem or a case. In contrast to that, this
article presents analytical solutions of laminar (exact) and turbulent thermal boundary layer equations and
formulate general features of conjugate convective heat transfer. These analytical results differ in principle from
those gained by common methods based on heat transfer coefficient which in fact are empirical means.

The article is organized in five sections following the introduction which consists of explanation of conjugation
idea and short literature review. The first section presents general expression for heat flux on streamlined body
surface with arbitrary temperature distribution obtained by solutions of laminar (exact) and turbulent thermal
boundary layer equations. In section 2, on the basis of this universal expression, the charts for conjugate solution of
simple heat transfer problems in laminar and turbulent flows are developed. Eight examples of solution of the
conjugate heat transfer problems under different boundary conditions demonstrate how much the modern strong conjugate solutions differ from data gained by common approach based on heat transfer coefficient. Analysis of conjugate results reveals the basic features of heat transfer process and the role of various parameters in determining the intensity of the conjugate heat transfer. In the next three sections, the properties observed in particular problems are generalized using presented in the first section universal expression for heat flux on a streamlined body surface with arbitrary temperature distribution. The effect of temperature head variation is analyzed in section 3, indicating that this issue is the major conjugate heat transfer characteristic. In section 4, it is shown that Biot number defining the ratio of body-fluid thermal resistances is the second parameter largely specified the heat transfer intensity. The final section 5 considers the universal expression for heat flux on a body surface as general boundary condition from which the known particular forms follow. It is shown that the estimation of the second term of this general condition with the first derivative of temperature head gives understanding whether or not the conjugate solution is required for some specific heat transfer problem. In the conclusion, a question “Should any heat transfer problem be considered using conjugate approach?” is discussed. As a whole, the article presents the modern conjugate heat transfer method for studying convective heat transfer substituting empirical approach based on heat transfer coefficient.

**Effect of Nonisothermicity**

Indeed, the conjugate heat transfer problem considers the thermal interaction between a body and a fluid flowing over or inside it. As a result of such interaction, a particular temperature distribution establishes on the body-fluid interface. This temperature field determines the heat flux distribution on the interface and virtually defines the intensity and properties of conjugate heat transfer. On the other hand, it is obvious that these properties of conjugate heat transfer and those of heat transfer from some nonisothermal surface are the same if in both cases, the temperature distributions are identical no matter how this distribution arose, as a result of conjugate procedure or by some other ways. Those considerations show that the question of conjugate heat transfer may be considered as a problem of heat transfer from arbitrary nonisothermal surface. From arbitrary because the temperature on the interface in typical conjugate problem is unknown in advance.

Such approach is realized in the author studies [25] and [26] using solutions of thermal boundary layer equation for laminar and turbulent flows. It is shown that the exact solution of a laminar thermal boundary layer equation for the plate with an arbitrary temperature distribution may be presented as a series of the consecutive derivations of the temperature head \( \theta_w = T_w - T_\infty \)

\[
g_w = h_w \sum_{k=1}^{\infty} g_k x^k \frac{d^k \theta_w}{dx^k}
\]

where \( h_w \) is heat transfer coefficient on an isothermal surface. Coefficients \( g_k \) for laminar and turbulent flows rapidly decrease with the increasing numbers, so that the series converges fast, and retaining two or three first terms usually yields satisfactory results. For laminar flows at zero pressure gradient, only coefficient \( g_1 \) depends on Prandtl number for small values (\( \Pr < 0.5 \)), while the next three coefficients in the whole range of Prandtl number (0 < \( \Pr < \infty \)) as well as the first coefficient \( g_1 \) for moderate and large values (\( \Pr > 0.5 \)) are practically constant and equal to corresponding values for the case of \( \Pr \to \infty \): \( g_1 = 0.6123, g_2 = -0.1345, g_3 = 0.03, g_4 = 0.006. \)

For turbulent flows and zero pressure gradient the coefficients \( g_k \) depend on Prandtl and Reynolds numbers significantly decreasing with increasing both \( \Pr \) and \( \Re \). They are smaller than those for laminar flows, and their maximum values are: \( g_1 = 0.5, g_2 = -0.05, g_3 = 0.01, g_4 = -0.01 \) for \( \Pr = 0.01 \) and \( \Re_{\phi^*} = 10^4 (\Re_x \approx 3 \cdot 10^4) \). For large Reynolds numbers \( \Re_{\phi^*} > 10^9 (\Re_x > 2.5 \cdot 10^2) \), the largest coefficient is \( g_1 \approx 0.1 \), and for \( \Pr > 100 \) all coefficients \( g_k \approx 0 \), leading to negligible effect of nonisothermicity. Here, \( \Re_{\phi^*} \) is the Reynolds number defined through displacement thickness \( \delta^* \) (more details in [24]).
Charts for Solving Simple Conjugate Problems

We begin from the basic classical problem of heat transfer between two fluids separated by a thin plate which imitates a heat exchanger. In common approach, the temperature head at one wall side is determined via overall coefficient $h_\Sigma$ as follows

$$\theta_{w1} = \frac{T_{x1} - T_{x2}}{T_{x1} - T_{x2}} = \frac{q_w}{h_\Sigma (T_{x1} - T_{x2})} = \frac{1}{1 + h_{s1}/h_{s2} + h_{s2} \Delta/\lambda_w}$$

where $\Delta$ and $\lambda_w$ are the plate thickness and conductivity. In what follows, the conjugate solutions of the same problem for the plate at different thermal boundary conditions show how strong the correct results differ from those giving by old simple equation (3) still presented in college courses as a key approach in convective heat transfer.

Consider a thermally thin plate streamlined by two fluids with temperatures $T_{x1}$ and $T_{x2}$. The term thermally thin means that the transverse thermal resistance of the thin body is small in comparison with resistance of a fluid, and due to that, the temperature across the plate may be considered as practically constant. In this case, a two-dimensional conduction equation after integration across the plate thickness simplifies to the form

$$\frac{d^2 T_w}{dx^2} = \frac{q_{w1} + q_{w2} + q_{v,av}}{\lambda_w \Delta} = 0, \quad \theta = \frac{T_{w1} - T_{x1}}{T_{x2} - T_{x1}}, \quad \xi = x / L, \quad \text{Bi}_{sL} = \frac{h_{sL} L^2}{\lambda_w \Delta}$$

Substituting series (2) with only the first three terms for heat fluxes $q_{w1}$ and $q_{w2}$ into this equation after applying variables (4) yields the differential equation determining the plate temperature [27]

$$D_k \theta + D_{01} \xi \theta' + \left(D_{02} \xi^2 - \xi^{r/s}\right) \theta' + \text{Bi}_{sL} = \frac{\hat{q}_{v,av} L^2}{\lambda_w (T_{x2} - T_{x1})}, \quad h_{\Sigma} = \frac{h_{sL} (x/L)^{-r/s}}{\lambda_w}$$

Here, $\hat{q}_{v}$ is the dimensional average value of volumetric heat source, indices 1, 2 indicate the plate sides, and $\text{Bi}_{sL}$ is a special form of Biot number which appears in derivation of equation (5), and which is defined as a product of usual Biot number $h_{sL} L / \lambda_w$ and ratio $L / \Delta$. In obtaining equation (5) from the first relation (4), the following issues are taken into account: (i) the sum of dimensionless temperature heads defined by the second equation (4) is $\theta_{w1} + \theta_{w2} = \theta_1 + \theta_2 = \theta$, and due to that equation (5) contains only one function $\Theta$, (ii) the Biot numbers for both plate sides are different if the heat transfer isothermal coefficients $h_{s}$ for both fluids are different, and therefore, two Biot numbers are present in equation (5) (iii) the exponent in the last formula (5) is given as a fraction $r/s$ because its value depends on flow regime, pressure gradient and rheology type of fluid. For example, this exponent is $1/2$ for laminar and $1/5$ for turbulent flows, $(1 - m)/2$ for self-similar flows with external velocity $U = C x^m$ [28], and its value also depends on exponent $n$ for power law non-Newtonian fluids (see notes to Table 3).

Here, we consider the relatively simple case when the pressure gradient is zero ($m = 0$), and flow regimes are the same on both sides of a plate resulting in only one exponent $r/s$ containing in equation (5). Nevertheless, there are two Biot numbers in this equation because the isothermal heat transfer coefficients $h_{sL}$ at the plate end for both streams are different if the velocities or physical properties of both streams are not the same leading to different Reynolds or/and Prandtl numbers. Other solutions of similar more complicated conjugate problems are considered in book [24].

In the simple case considered here, the equation (5) is transformed by employing a new variable $z$ instead of $\xi = x / L$ to the form
\[ \theta + g_1 z \frac{d\theta}{dz} + (g_2 z^2 - \frac{r}{s}) \frac{d^2 \theta}{dz^2} - \sigma_{Bi} \frac{q_v}{z^L} = 0 \]  

\[ z = z_L \left( \frac{x}{L} \right), \quad z_L = \left( \frac{Bi_{sL1} + Bi_{sL2}}{2} \right)^{2-r/s} \quad \sigma_{Bi} = \frac{Bi_{sL2}}{Bi_{sL1} + Bi_{sL2}} \]

It is seen that the plate temperature defined by equation (6) depends only on one variable \( z \). Equation (6) also does not contain any boundary conditions of a particular problem. Therefore, function \( \theta(z) \) is tabulated and used for solving different conjugate heat transfer problems. In order to distinguish the tabulated functions from others, we apply for these a special notation \( \phi(z) \).

Four functions for laminar and turbulent flows are tabulated for some simple constant boundary conditions: \( \phi_1 \) and \( \phi_2 \) giving the solutions of homogeneous (without a heat source) equation (6) and \( \phi_3 \) and \( \phi_4 \) assigning particular solution of inhomogeneous equation (6) for the case of uniform or linear heat source \( q_v = A + B \left( \frac{x}{L} \right) \). The homogeneous equation (6) is modified to well-known hypergeometric equation. Then, the functions \( \phi_1 \) and \( \phi_2 \) are determined by two hypergeometric functions

\[ \phi_1 = F \left( \alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, z^{2-r/s} \right), \quad \phi_2 = z F \left( \alpha, \beta, \gamma, z^{2-r/s} \right) \]

where \( \alpha + \beta = (1 + g_1 / g_2) / (2 - r/s) \), \( \alpha \beta = (1 + g_1 / g_2)^2 / (2 - r/s) \) so that \( \alpha \) and \( \beta \) are roots of quadratic equation and the third parameter is defined as \( \gamma = (3 - r/s) / (2 - r/s) \). The following simple boundary conditions are used for tabulating functions

\[ \phi_1(0) = 1, \quad \phi_2(0) = 0, \quad \phi_3(0) = 0, \quad \phi_4(0) = 1 \]

For the problems with thermal sources \( q_v \), the temperature head is presented as a sum of a general \( \phi_1 \) and \( \phi_2 \) and a particular \( \sigma_{Bi} + \phi_4 \) solutions of equation (6).

\[ \theta = C_1 \phi_1 + C_2 \phi_2 + \sigma_{Bi} + \phi_4 \quad \phi_4 = A \phi_3 + B \phi_4 \]

\[ A = \frac{\lambda L^2}{\lambda w (T_{x=2} - T_{x=1}) z_L^2}, \quad B = \frac{\dot{\sigma} L^2}{\lambda w (T_{x=2} - T_{x=1}) z_L^3} \]

The constants \( C_1 \) and \( C_2 \) in (9) are found using boundary condition given in the problem in question. The dimensionless local and total heat fluxes from a plate, and longitudinal heat flux along the plate are expressed via derivatives of functions (9) \( \theta \) defined in terms of tabulated derivatives \( \phi_1' \) and \( \phi_2' \)

\[ q_w = \frac{\dot{q}_w L^2}{\lambda w \Delta T_R z_L^3 \Delta} = \frac{\theta'}{20 \theta}, \quad Q_w = \frac{2 L \dot{Q}}{\lambda w \Delta T_R z_L^3} = \frac{\theta'(z_L) - \theta'(0)}{\theta_0} \]

\[ q_x = -\frac{\dot{q}_x L^2}{\lambda w \Delta T_R z_L^3} = \frac{\theta'}{\theta_0}, \quad \chi_l = \frac{h}{h_*} = 1 + g_1 x \frac{\theta'}{\theta} + g_2 x^2 \frac{\theta''}{\theta} \]

The last equation (10) determines the nonisothermicity coefficient which shows how much the heat transfer coefficient \( h \) gained in conjugate solution differs from isothermal coefficient \( h_* \). Here, this coefficient is estimated applying only the first three terms of series (2) knowing that coefficients \( g_k \) for \( k > 3 \) are negligible small. In expressions (10), the dimensional values are marked by “cap”, \( \Delta T_R = T_R - T_\infty \) is the difference of reference temperature \( T_R \) and index 0 denotes values at the leading edge.
For laminar and turbulent flows, functions $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$ and their derivatives are presented in Figures 1-4.

For laminar flow, the tabulated functions are valid for zero pressure gradients and $Pr > 0.5$ for which the coefficients $g_k$ are independent of Prandtl number. For turbulent flow, the tabulated functions are applicable for zero pressure gradients, $Pr \approx 1$ and $Re_x = 10^6 - 10^7$. Examples of solutions with computing details and analyses are considered below. Some others may be found in [24].
Example 1 A steel plate of length $0.25\text{m}$ and of thickness $0.01\text{m}$ is in a symmetrical air flow of velocity $3\text{m/s}$ and of temperature $300\text{K}$. The left-hand end is insulated, and the temperature of the other end is maintained at $T_w$. Because in this case, the temperature of the plate end $T_w(L)$ is specified only, it is reasonable to take it as a reference temperature, and use dimensionless temperature head in the form $	heta = (T_w - T_u)/(T_u - T_w)$. Applying boundary conditions $\theta(L) = 1$ for the right end, and $q_x(0) = 0$ for the isolated left end, one finds from first relations (9) and third relation (10) two equations $C_1\theta(z) + C_2\theta(z) = 1$ and $C_3\theta(0) + C_4\theta(0) = 0$, respectively. Then, from the second equation follows that $C_1 = 0$ because according to (8) $\theta(0) = 1$ and $\theta(0) = 0$. Due to that, the first equation gives $C_2 = 1/\theta(z)$. Since $Re = 5 \cdot 10^4$, the flow is laminar, and the exponent in relations (6) is $r/s = 1/2$. Using this value, we estimate the Biot number (4)

$$Bi = \frac{hL^2}{\lambda L} = \frac{NuL^2}{\lambda L} = \frac{29.5 \cdot (5 \cdot 10^4)^{1/2} \cdot 0.278 \cdot 0.25}{0.65} = 0.66,$$

the third parameter (6) $z_L = (2 \cdot 0.66)^{1/3} = 1.2$. Finally, the temperature head and heat fluxes are defined by relations (9) and (10) $\theta = C_3\theta(z)/\theta(z)$ and $q_x = -\theta(0)/\theta(z)$. The required values of tabulated function and their derivatives are taken from Figure 1. The numerical results are given in Table 1.

<table>
<thead>
<tr>
<th>$z$</th>
<th>$x/L$</th>
<th>$\theta(z)$</th>
<th>$\theta'(z)$</th>
<th>$\theta''(z)$</th>
<th>$\theta$</th>
<th>$-q_x$</th>
<th>$q_w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\infty$ $\approx 0$</td>
<td>$0.278$</td>
<td>$0$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>0.2</td>
<td>0.167</td>
<td>1.12</td>
<td>0.949</td>
<td>2.13</td>
<td>$0.311$</td>
<td>$0.264$</td>
<td>$0.382$</td>
</tr>
<tr>
<td>0.4</td>
<td>0.334</td>
<td>1.37</td>
<td>1.48</td>
<td>2.56</td>
<td>$0.388$</td>
<td>$0.411$</td>
<td>$0.369$</td>
</tr>
<tr>
<td>0.6</td>
<td>0.501</td>
<td>1.72</td>
<td>2.04</td>
<td>3.02</td>
<td>$0.478$</td>
<td>$0.567$</td>
<td>$0.429$</td>
</tr>
<tr>
<td>0.8</td>
<td>0.668</td>
<td>2.19</td>
<td>2.70</td>
<td>3.24</td>
<td>$0.608$</td>
<td>$0.750$</td>
<td>$0.503$</td>
</tr>
<tr>
<td>1.0</td>
<td>0.835</td>
<td>2.81</td>
<td>3.50</td>
<td>4.42</td>
<td>$0.780$</td>
<td>$0.969$</td>
<td>$0.614$</td>
</tr>
<tr>
<td>1.2</td>
<td>1</td>
<td>3.60</td>
<td>4.48</td>
<td>5.12</td>
<td>$1$</td>
<td>$1.244$</td>
<td>$0.753$</td>
</tr>
</tbody>
</table>
The data from Table 1 are valid in both cases of increasing and decreasing temperature head because they are given in dimensionless values \((T_w - T_s)/(T_{\infty} - T_s)\). To see the different effects of temperature head variation in those two cases, we compute the nonisothermicity coefficient using the last formula (10) with three terms of series (2). This procedure requires the relation for temperature head found above \(\theta = \theta'\left(z\right)/\theta'\left(z_L\right)\) and two similar formulae for derivatives \(\theta' = \theta''\left(z\right)/\theta'\left(z_L\right)\) and \(\theta'' = \theta'''\left(z\right)/\theta'\left(z_L\right)\). Using data from Table 1, we get: 

- For growing (positive) temperature head: \(\chi = 1.09, 1.22, 1.35, 1.46, 1.55, 1.64\)
- For falling (negative) temperature head: \(\chi = 0.909, 0.777, 0.649, 0.538, 0.449, 0.361\)

It is seen that the temperature head variation strongly affects the heat transfer intensity. While in the case of the increasing temperature head, the heat transfer coefficient increases and reaches at the end of the plate a value of about 65% greater than the isothermal coefficient \((\chi = 1.64)\), in the opposite case of falling temperature head, it strikingly decreases and becomes almost three times less than isothermal coefficient at the terminal edge \((\chi = 0.361)\). If in this case the plate was longer, the heat flux and heat transfer coefficient further would approach zero and then would change their directions leading to heat inversion—the phenomenon which is similar to separation.

As we will see from more following examples, such strongly different effects of increasing and decreasing temperature heads on the heat transfer coefficient values are the general properties of heat transfer. More precisely and detailed, we discuss this and other general properties of heat transfer in final sections and conclusion.

**Comment 1** Here and below, we analyze chiefly the case of positive temperature head when the body temperature \(T_s\) is higher than that of the fluid \(T_{\infty}\). In the case when the body is cooler than the fluid, and the temperature head is negative \((T_s < T_{\infty})\), these results are valid if one considers the temperature head as a negative value. Otherwise, if the absolute value of temperature head is employed in the second case, the opposite results are valid: a decreasing temperature head leads to growing heat transfer coefficient, and increasing temperature head results in falling heat transfer coefficient in comparison with that for an isothermal surface. That is because the derivative \(\partial \theta / \partial x\) in the second term of relation (2) and last equation (10) changes the sign to the opposite in the case of negative temperature head. However, the same conclusions are valid in both cases if one considers in the case of negative temperature head it absolute value.

**Example 2** Air at a temperature \(313 K\) flows with velocity \(30 m/s\) over one side of a copper plate \(0.5 m\) long and \(0.02 m\) in thickness. Another side of the plate is isolated. The temperatures of the leading and trailing ends are \(T_{w0} = 593 K\) and \(T_{wL} = 293 K\).

If the dimensionless temperature head is defined as \(\theta = (T_w - T_s)/(T_{\infty} - T_s)\), the boundary conditions are \(\theta(0) = 1\) and \(\theta(L) = \theta_L\). According to the first relation (9), these conditions give two equations \(C_1 \theta(0) + C_2 \theta_L(0) = 1\) and \(C_1 \theta_L(\zeta_L) + C_2 \theta(\zeta_L) = \theta_L\), which along with conditions (8) yielded from first equation \(C_1 = 1\) and then from the second one \(C_2 = \theta_L - \theta(\zeta_L)\). The Reynolds number \(Re = 0.88 \cdot 10^6\) tells us that flow is turbulent, and hence, \(r/s = 1.5\), and \(Nu = 0.0255 Re^{0.8}\). Then, \(Bi_L = 2.53\), \(z_L = 1.67\), and the temperature head according to (8) and just found \(C_1\) and \(C_2\) is \(\theta = \theta_L(z) + \frac{\theta_L(z) - \theta_L(\zeta_L)}{\theta_L(z) - \theta_L(\zeta_L)} \theta_L(z)\). Finally, after taken functions \(\theta\) from Figure 2, one finds \(\theta = \theta_L(z) - 1.22 \theta_L(z)\). The results are plotted in Figure 5 that shows the same dependence for the heat transfer coefficient the value of which intensively goes down, reaches zero close to the plate end and then the heat flux inversion occurs. The reason of such results as indicated above is the decreasing temperature head along a plate.
Example 3 Consider the same problem for an aluminum plate of length 0.3 m and with thickness of 0.002 m streamlined by a flow of air with velocity 250 m/s on an altitude of 20 km. Air temperature is $T_\infty = 223$ K, a kinetic viscosity is $\nu = 1.65 \cdot 10^{-4}$ m$^2$/s. The front end temperature is a stagnation temperature $T_{w0} = T_{x0} = 254$ K, and the other end is maintained at $T_{wL} = 323$ K.

Because the Mach number in the problem in question is relatively high $M = U/a = 250/20.1 \sqrt{223} = 0.833$ (here, $a = 20.1 \sqrt{223}$ is the speed of sound), it is necessary to take into account the effect of compressibility. It is known that in the case of not very high Mach number, this may be done by employing the adiabatic temperature $T_{ad}$ instead of temperature $T_\infty$. Therefore, in this case, the dimensionless temperature head is present in the form $\theta = (T_w - T_{we})/(T_{ad} - T_{we})$ where $T_{wL}$ is used as a reference temperature and the stagnation (adiabatic) temperature $T_{w0} = T_{x0}$ is applied instead of $T_\infty$. Such form of a temperature head provides simple boundary conditions: $\theta(0) = 1$ and $\theta(L) = 0$. Then, in similar way by using equation (9) for $\theta$ along with relation (8), one gets two relations and defines constants $C_1 = 1$ and $C_2 = -\theta_L(z_L)/\theta_L(z_L)$ and after substituting constants into (9) obtains the solution $\theta = \theta_L[z] - [\theta_L(z_L)/\theta_L(z_L)]$. Reynolds number $Re = 4.55 \cdot 10^5$ tells us that the flow is laminar giving $Nu_x = 200$ and $Bi_{wL} = 2.91$. Using these values and taking data for tabulated function from Figure 1, the last equation for temperature head is transformed to the form $\theta = \theta_L(z) - 1.60 \theta_L(z)$. The results for heat fluxes are gained from equations (10) after substituting the difference $(T_{x0} - T_{wL})$ for $(T_w - T_\infty)$. The numerical data presented in Table 2 indicate that in this case, the heat flux inversion ($q_w = 0$) occurs close to plate end, at $x/L = 0.783$, leading farther to negative heat fluxes.

<table>
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<th>$z$</th>
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<th>$\theta_L(z)$</th>
<th>$\theta_L(z)$</th>
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Example 4 Heat transfer from the plate heated from one end

This example shows clear-cut the role of temperature head variation because the heat transfer characteristics of such a plate significantly differ depending on the flow direction in two cases: from heated to unheated ends (the first case) and at an opposite direction, from unheated to heated ends (the second case). If one considering this
Abram Dorfman

problem ignores the interface temperature head distribution, there would not be any differences in heat transfer characteristics for both flow directions which is incorrect.

To see the real situation, consider the last equation (10). It is clear that in the first case, when the flow starts from a heated edge, the temperature head decreases in the flow direction. Hence, in this case \( \frac{d\theta}{dx} < 0 \), and according to relation (10) the ratio (nonisothermicity coefficient) \( \chi = \frac{h}{h^*} \) decreases along the plate. At the same time, the heat transfer coefficient of the isothermal surface \( h^* \) also lessens along the plate. Thus, the overall heat transfer coefficient \( h \) defined as a product \( \chi h^* \) strictly decreases along the plate. An opposite situation takes place in the second case when the flow starts from the coolest edge. In this situation, the temperature head increases in flow direction giving \( \frac{d\theta}{dx} > 0 \) so that according to the same equation (10), the product \( \chi h^* \) grows along the plate.

Since in this case, the heat transfer coefficient \( h^* \) for an isothermal surface decreases along the plate as well, the overall heat transfer coefficient \( h = \chi h^* \) decreases or increases depending on ratio of the two multiplying factors in the last equation. This physical analysis is proved by following conjugate solution.

Let the temperature head of the heated end is \( \theta_h \), and the other end is isolated. Determining the temperature head for the case of symmetrical flow as \( \theta = (T_n - T_c)/T_n \), we have the following boundary conditions: \( \theta(0) = \theta_h \) and \( \theta'(L_z) = 0 \) in the first case and \( \theta'(0) = 0 \) and \( \theta'(L_z) = \theta_h \) in the second case. These conditions give: \( C_1 = \theta_h \) and \( C_2 = -\theta_h \frac{\delta_1(z)}{\delta_1(z)} \beta_1(z) \) and \( C_2 = 0 \) and \( C_1 = \theta_h / \beta_1(z) \) for the first and second cases, respectively. Then, employing equations (9), (10) and (8) leads to the temperature heads, heat fluxes and gives the ratio of total heat fluxes in both cases

\[
\frac{\theta}{\theta_h} = \frac{\delta_1(z)}{\delta_1(z)} \beta_1(z), \quad \frac{\theta}{\theta_h} = \frac{\delta_1(z)}{\beta_1(z)}, \quad \frac{Q_1}{Q_{n-1}} = \frac{\delta_1(z)}{\beta_1(z)} \]

The results for laminar flow are plotted in Figure 6. It is seen that heat transfer characteristics differ substantially in both cases. In the first case when the temperature head decreases, the heat transfer coefficients are significantly less than the isothermal coefficients, and the heat flux sharply decreases along the plate, so that the situation is close to inversion at the plate end. Here, the heat transfer coefficient is 4.5 times less than an isothermal coefficient. In the second case, the temperature head increases in flow direction, and according to that, the heat transfer coefficients are greater than the isothermal ones, but not more than 1.8 times. Nevertheless, the total heat flux in this case is less than that in another case with the decreasing heat transfer coefficients. This seemingly strange outcome happened because in the first case there are large temperature heads and heat transfer coefficients at the start of flowing, while in the second case at the beginning when the heat transfer coefficient are large, the temperature heads are small and vice versa. As a result, the local heat flux has the minimum in this case in contrast to that for the first case (curves 3 in Fig. 6).

FIGS.6 HEAT TRANSFER CHARACTERISTICS FOR THE PLATE HEATED FROM ONE END. a) LOCAL CHARACTERISTICS \( \text{Bi}_{L_1} = 1.4, 1 - \text{FIRST CASE, II - - - - - - SECOND CASE,} 1 - 2\eta, 2 - \theta / \theta_h, 3 - \chi^* \); b) RATIO OF TOTAL HEAT FLUXES REMOVED FROM PLATE 1- TUEBULENT FLOW AND 2-LAMINAR FLOW.
The value of the ratio of total heat fluxes $Q_{w1}/Q_{w2}$ depends on Biot number and in the case of laminar flow riches significant values (Fig. 6(b)). For instance, for steel plate with $\Delta / L = 1/10$ past air ($\text{Bi}_L = 0.8$) and water ($\text{Bi}_L = 4.5$), this ratio is 1.2 and 1.65, respectively. In the case of turbulent flow, the difference between total fluxes is smaller (Fig. 6(b)), but the distributions of the local heat fluxes along the plate in two opposite directions differ in essence as well which is easy to check using presented charts for conjugate problems solution.

**Comment 2** This example is a model of a situation when a heated from one edge object is cooled moving through surroundings of air or water. If the heat is supplied at a leading edge, the temperature head decreases in flow direction, whereas if the heat is delivered through the trailing edge, the temperature head increases in flow direction as in the first and the second cases, respectively, in the just considered model. Since the removed total heat in the first case is greater than that in the second one, the average temperature of the body with leading heat source is less than in the case with the same heat source located at the trailing edge. As indicated above, this difference for large Biot numbers reaches 1.5-1.6 for laminar and about 1.2 for turbulent flows. The reason of this as it was revealed many times in this text is an entirely different effect of the temperature head variation resulting in substantial contrariety in heat transfer coefficients (curves 3 in Fig. 6).

**Example 5** Heat transfer from a plate in flow past one side and isolated at another. Effect of initial heat fluxes

Let the temperature $T_0$ and the heat flux $q_0$ are given at the starting end of the plate. Using these boundary conditions, equation (9), the third equation (10) and conditions (8), we find the constants $C_1 = \theta_0, C_2 = \theta_0' = q_0\theta_0$ (since according to (10) $q_0 = \theta_0'/\theta_0$). Then, the solution for the temperature head is obtained as

$$\theta = \theta_0 \left[ \vartheta_1(z) + q_0\vartheta_2(z) \right]$$

(13)

Figure 7 shows the variation of the nonisothermal coefficient $\chi_t$ and the temperature head for laminar (a) and turbulent (b) flows for the three cases: $q_0 = 10.0$ and $(-2)$.

In the first two cases, the temperature head increases along the plate, while in the third one, the temperature head first decreases and after reaching zero, its absolute value starts to increase. The same character of heat transfer variation as in other examples is observed. For an increasing temperature head, the heat transfer coefficients are greater than those for an isothermal surface but not more than 75% to 80% in the case of laminar flow and not more than 20% to 25% for the turbulent flow. In the third case in which the temperature head decreases, these coefficients are so much smaller that in some points where temperature head turns to zero, the heat transfer coefficient becomes meaningless (as it first was explained in [30]), and a curve $\chi_t(z)$ undergoes discontinuity (3 on Fig. 7).

**Example 6** Heat transfer from a plate in flow past one side and isolated at another. Effect of boundary conditions
Here, we consider the same problem but with different boundary conditions. It is shown that the solution just obtained for boundary conditions specified at the leading edge is valid for some cases with other boundary conditions. In particular, if conditions are given at the terminal end, this is achieved by estimating an equivalent value of the leading heat flux $q_0$. For two cases when at the terminal end the temperature $T_L$ or the heat flux $q_L$ is specified instead of $q_0$, the problem solution is the same relation (13) as in the previous example if one uses for the starting value $q_0$ the two following expressions

$$q_0 = \frac{\theta_L}{\theta_0 - \theta_L} \left( \frac{z_L}{\theta_L} \right), \quad q_0 = \frac{q_L - \theta_L'}{\theta_L'} \left( \frac{z_L}{\theta_L'} \right)$$

(14)

It is clear that in these cases, the heat transfer characteristics are the same as given on Figures 7 if the equivalent values $q_0$ are the same or may be obtained for other values of $q_0$ by approach described in Example 6.

**Example 7** Heat transfer from plate in flow past two sides at different fluid temperatures.

Since there are two flows, the second formula (4) for the temperature head with the scale in the denominator $T_{w2} - T_{w1}$ should be employed. Assuming that temperature head at the leading edge for both sides of the plate equals to the difference of given flow temperatures $T_{w2} - T_{w1}$, we have $\theta_0 = 1$ and obtain expressions for heat fluxes along the plate and from both sides of the plate

$$q_x = \theta', \quad q_{w1} = \theta' + \sigma_{Bi} z^{-r/s}, \quad q_{w2} = q_{w1} - z^{-r/s}$$

(15)

The first equation is found from the third relation (10) knowing that $\theta_0 = 1$. The second equation follows from equation (6) (with $q_v = 0$) because the first three terms in this equation determine the heat flux $q_{w}$, like in general case, the heat flux is determined by full series (2). The third equation (15) is obtained from the first equation (4) after transforming it to variable (6) $z = z_L(x/L)$ which is used for tabulated function. Then, solving the transforming equation for $q_{w2}$ and applying the result (15) for $q_{w1}$ gives the third equation (15).

Calculation was performed for turbulent flow, equal thermal resistances of both fluids $(Bi_{w1} = Bi_{w2}, \sigma_{Bi} = 0.5)$ and for the value of heat flux at starting edge $q_0 = -2$. Using this data, $\theta_0 = 1$ and the first equation (15) along with conditions (8), one finds the constants in equation (9) employing the same approach as in problems considered before. Then, taken the values of tabulated functions and their derivatives from Figure 2, the temperature heads for both sides are estimated regarding that $\theta_1 + \theta_2 = 1$ (see note (i) after equation (5)). Heat fluxes are obtained applying two last equations (15).

Figure 8 presents the results. The same pattern of heat transfer characteristics is observed: on the side with increasing temperature head, the nonisothermicity coefficient is a little more than unity (curve 2), while on the other side where at the beginning the temperature head decreases (curve 1), the heat transfer coefficient and the corresponding nonisothermicity coefficient $\chi_t$ sharply fall and after reaching zero go to $\pm \infty$ (curve 1) becoming discontinuous and meaningless as it is explained in previous examples.

**Example 8** Heat transfer from a plate with inner heat sources.

The solution of this problem is found as a sum of general and particular solutions giving by function $\mathcal{G}_1$ and $\mathcal{G}_2$ for general part, as for other problems, and defined by functions $\mathcal{G}_3$ and $\mathcal{G}_4$ for the particular part of solution for uniform or linear sources. Calculations are performed for turbulent flow considering two cases: past one side and past two sides of a plate with linear heat source. The following conditions are specified: $\theta_0 = 1, q_0 = 0, A = 1$ and $B = 2$ (see equations (9)). Figure 9 shows that on one side of dual streamlined plate where the negative temperature head increases (dashed curve 2), the heat transfer coefficients do not much differ from those for an isothermal surface (the nonisothermicity coefficients do not much differ from unit). For one side streamlined plate and another side of dual streamlined plate, the temperature heads decrease (dashed curves 1 and 3) and the effect
of nonisothermicity is as significant as in the other cases with decreasing temperature heads despite of the turbulent flow regime.

![Diagram of Temperature Head and Nonisothermicity Coefficient](image1)

**FIG. 8 VARIATION OF TEMPERATURE HEAD AND NONISOTHERMICITY COEFFICIENT ALONG THE PLATE STREAMLINED ON BOTH SIDES BY TURBULENT FLOW, $\sigma_{Bi} = 0.5, \theta_0 = 1, \bar{\sigma}_0 = -2$, ------ $\chi_\theta$.**

$\cdot \cdot \cdot \theta_1, 1, 2$ DIFFERENT SIDES OF A PLATE

![Diagram of Heat Transfer Characteristics](image2)

**FIG. 9 HEAT TRANSFER CHARACTERISTICS FOR THE PLATE WITH INNER HEAT SOURCES STREAMLINED BY TURBULENT FLOW, ---- $\chi_\theta, \cdot \cdot \cdot \theta_1, 2$ DIFFERENT SIDES OF A PLATE, $\sigma_{Bi} = 0.5, 3$ ONE SIDE STREAMLINED PLATE ($\sigma_{Bi} = 0$)**

**Temperature Head Variation-Major Conjugate Heat Transfer Characteristic**

Presented analysis of different particular problems demonstrated how much the conjugate method differs from a common approach showing the essential role of temperature head variation (nonisothermicity effect) along the body-fluid interface, which is ignored in common means. Proceeding from this point, we formulate below the general properties of heat transfer that follow from general expression (2) for heat flux on a body surface (which is body-fluid interface) at arbitrary temperature head distribution.

Because the expression (2) presents the heat flux as a series of successive derivatives of the temperature head, this relation is used for investigating the effect of temperature head distribution on the heat transfer intensity. Physically, such series can be considered as a sum of perturbations of the uniform surface temperature. The case when all derivatives are zero, and in series (2) only the first term $h_\theta \theta_0$ retains, corresponds to an isothermal surface which may be considered as undisturbed temperature head. The series with two first terms containing the only first derivative presents the linear disturbed temperature field. The series consisting of two derivatives describes the quadratic distributed temperature field, and so on, finally resulting in series with infinite terms giving the arbitrary temperature field distribution.
Calculation shows [24] that the coefficients $g_k$ of series (2) rapidly decrease. Due to that, the contribution of each series term diminishes as the number of derivative grows. Moreover, comparison shows that the first coefficient $g_1$ is significantly larger than others in all studied cases, and even the largest among others, the second coefficient $g_2$, is less than the first one from three to ten times in different cases (see Table 3). This result indicates that the first derivative (i.e., temperature head gradient) basically defines the effect of the surface temperature head variation or in other words, the effect of the nonisothermicity of the body surface (or body-fluid interface).

### Table 3 Relation between Coefficients $g_1$ and $-g_2$

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Note to Table 3: the numbers on the top indicate the flow type and conditions associated with obtained data: Laminar layer: arbitrary $\theta_\infty \rightarrow 0$, $2-Pr \rightarrow \infty$, arbitrary $q_-\rightarrow 3-Pr \rightarrow \infty$, $4-Pr \rightarrow 0$, unsteady laminar thermal layer $5-Pr=1$, turbulent layer: $6-Pr \rightarrow 0$, $Re_{\delta^*}=10^3$, $7-Pr \rightarrow 0$, $Re_{\delta^*}=10^3$, $8-Pr=1$, $Re_{\delta^*}=10^3$, $9-n=1.8$, $10-n=0.2$, $11-Pr \approx 1$, moving surface.

We have seen from considered examples that increasing and decreasing temperature heads strongly different affect the value of heat transfer coefficient. While in the case of growing temperature head, the heat transfer coefficient increases and reaches values moderately greater than that of isothermal surface, in the opposite case of falling temperature head, it decreases sharply becoming times less than isothermal heat transfer coefficient and may attain even zero and get negative resulting in inversion phenomenon.

More clear and assured, this can be seen from the general expression (2) or from the last relation (10) for nonisothermity factor. Because the first coefficient $g_1$ in these expressions is positive, the increasing temperature heads (positive gradients) always lead to an increase of heat transfer intensity, while the decreasing temperature heads (negative gradients) ever cause a decrease of the heat transfer intensity. Or more precisely: if the temperature head increases in the flow direction or in time, the heat transfer coefficient is greater than isothermal coefficient, whereas a decreasing of the temperature head along the flow direction or in time yields a decrease of the heat transfer coefficient compared with the isothermal one.

At the same time, a simple analysis shows that an identical change of increasing and decreasing temperature heads results in significantly different variations in the heat transfer coefficients. The reason of this is that the same absolute difference in the temperature head and in corresponding heat transfer coefficient yields much greater change in the values of relative variation in the cases of a falling than in a growing temperature heads. Moreover, as it can be seen from the same formulae (2) and (10) in the case of decreasing temperature head (negative first derivative), the heat flux and heat transfer coefficient becomes zero if the negative temperature gradient is large or surface is sufficiently long. Even in turbulent flows, where the nonisothermicity effects are relatively small, the falling temperature head may result in zero heat transfer coefficient. This is true for all Reynolds and Prandtl numbers except the case of turbulent flows with high Prandtl numbers (say more than 100), when the nonisothermicity effect is negligible.

The temperature head gradient plays the same role in heat transfer as the pressure gradient acts in flow (gradient analogy [24]). The favorable positive temperature head gradient acts similar to positive velocity (or negative pressure) gradient, while the unfavorable negative temperature head gradient works analogous to negative
velocity (or positive pressure) gradient. In the last case, in heat transfer may develop inversion of heat flux, when
the heat flux becomes zero (or heat transfer coefficient) and then changed its direction [29]. This phenomenon is
similar to flow separation occurring in flows with adverse negative velocity gradients. Despite the similarity, there
is a principle difference between those two processes: after separation, the flow changes its structure, so that
boundary layer does not exist anymore, whereas in heat inversion only direction of heat flux changes without
breaking the flow pattern.

Thus, the general expression (2) for heat flux (in conformity with conjugate solutions) shows that temperature head
(or temperature at $T_\infty = \text{const.}$) variation along the body surface basically determines the intensity and
characteristics of heat transfer acting as favorable variation in the case of increasing temperature head in flow
direction or in time and working as unfavorable variation usually resulting in singularities in opposite case when
the temperature head decreases in flow direction or in time.

**Biot Number as a Second Important Parameter**

The other parameter defined the intensity of conjugate heat transfer is the ratio of thermal resistances of the body
and flowing fluid. In the case of the given temperature head variation, this ratio largely specifies the absolute value
of heat transfer change caused by the surface nonisothermicity. This can be shown by using the same expression (2)
for heat flux or the last equation (10) for nonisothermicity coefficient along with the conjugate conditions. By taking
into account that, the nonisothermicity effect is basically determined by the second term of the series (2), one gets
from conjugate conditions (equalities of temperatures and heat fluxes on the interface) the following relation

$$\lambda_w \frac{\partial T}{\partial y} \bigg|_{y=0} = h \left( g_i \lambda w \frac{\partial T}{\partial x} \right) \text{ or } \frac{1}{\text{Bi}} \left( \frac{\partial T}{\partial (y/\Delta)} \right) \bigg|_{y=0} = g_i \lambda w \frac{\partial T}{\partial x}, \quad \text{Bi} \cdot \frac{h_\lambda}{\lambda_w} \tag{16}$$

The left side of equation (16) is the heat flux at the interface gained by using the Fourier law for a body, and the
right side of this equation is the second term of expression (2) (or of nonisothermicity coefficient (10)) determined
the heat flux at the interface in a fluid.

Equation (16) shows that the value of the temperature head gradient $\lambda w \frac{\partial T}{\partial y} y=0$ is inversely proportional to Biot
number for an isothermal surface which is a ratio of the thermal resistances of a body $(\Delta / \lambda_\infty)$ and a fluid $(1/h)$. It
follows from equation (16) that in both limiting cases $\text{Bi}_0 \rightarrow \infty$ and $\text{Bi}_0 \rightarrow 0$, the corresponding conjugate problem
degenerates, because in these cases, only one resistance is finite, while another either is infinite as in the first case or
becomes zero as in the second one. In the first case, a conjugate problem transforms into a problem with isothermal
surface, because according to (16) the temperature gradient in this case is zero. This case corresponds to a situation
when the body of infinite thickness (or negligible conductivity) is streamlined by the fluid with finite heat transfer
coefficient or when a body of finite conductivity and thickness is streamlined by the fluid with infinite heat transfer
coefficient. In the other case, a conjugate problem transforms into a problem of a body streamlined by a fluid that
changes temperature in a stepwise manner, because according to (16), the temperature head gradient is infinite in
this case. This case corresponds to a situation when the body of finite thickness and conductivity is streamlined by
the fluid with zero heat transfer coefficient, or the body with infinite conductivity (or negligible thickness) is
streamlined by fluid with finite heat transfer coefficient.

Because in both limiting cases, the conjugate problem decays, one concludes that the greatest effect of
nonisothermicity should be expected when the both resistances are of the same order, so that Biot number is close
to unity. The Biot number can be presented in various forms suitable for one or another particular conjugate
problem. In such a case, the Biot number characterizes the relation between the resistances of the body and fluid as
well, but the quantitative results can be different from unit. This is also the reason why there are several similar
other criteria characterizing the relation of body-fluid thermal resistances. For example, Luikov in an early work
suggested the Brun number $[31] \text{Br} = (\Delta/x)(\lambda / \lambda_\infty)(\text{Pe})^{1/3}$, or later Cole proposed similar criterion $(\lambda / \lambda_\infty)(\text{Pe})^{1/3}$
[32].

However, it is easy to see that those relations are, in fact, various Biot numbers.
General Boundary Condition and Common Approach Accuracy

Series (2) may be also considered as a sum of boundary condition perturbation [33]. The case when all derivatives are zero corresponds to the boundary condition with isothermal heat transfer coefficient which may be considered as undisturbed boundary conditions. The series containing only the first derivative presents the linear perturbed boundary condition. The series with two derivatives describes the quadratic perturbed boundary condition and so on. In a general case, the series consist of infinite number of derivatives and describes arbitrary boundary condition. From such considerations, one concludes that expression (2) can be treated as a general boundary condition describing different types of surface temperature distribution.

In the case of an isothermal surface, in series (2), it retains only the first term, and it becomes the boundary condition of the third kind. This boundary condition is still often employed despite existing effective numerical methods. Two reasons are responsible for that: the simplicity of his approach and the fact that there are some problems in which the nonisothermicity slightly affects the final results. In view of this fact, it is important to know an accuracy of common simple approach to see whether the conjugate solution is required. If the solution obtained by common approach using boundary condition of the third kind is known, the error caused by such approximation can be estimated by computing the second term of the general boundary condition (2). Comparing the value of the second term with the known approximate solution gives an understanding of common method accuracy and tells us whether a conjugate solution is required. Simpler the error may be estimated by computing the second term of expression (10) for the nonisothermicity coefficient which gives a value of relative error in fraction. Below we present some examples.

Example 9 Heat transfer from fluid to fluid in a flow past two sides of a thin plate.

In common approach, the temperature head at one wall side is determined via overall coefficient \( h_\Sigma \) by equation (3). If the flow regimes on both sides are the same, the ratio \( h_{11} / h_{22} \) in this equation does not depend on \( x \). Taking this into account and knowing that the Biot number for isothermal surface is a power-law function: \( \text{Bi}_n \sim x^{-n} \), one presents the denominator in equation (3) in the form \( D_1 + D_2 x^{-n} \) with constant coefficients. Then, equation (3) becomes

\[
\theta_{w1} = \frac{1}{D_1 + D_2 x^{-n}} \quad \sigma = \frac{g_1 x}{\theta_{w1}} \frac{d\theta_{w1}}{dx} = \frac{g_1 n D_1}{D_1 x^n + D_2} 
\]

As indicated above, the error \( \sigma \) caused by common approach due to ignoring the effect of nonisothermicity is estimated through the second term of the last equation (10) which in the case of the temperature head (17) is determined by second relation (17) obtained after differentiating the first expression (17) for \( \theta_{w1} \).

The maximum value of error (17) is \( \sigma_{\text{max}} = g_1 n \) at \( x = 0 \) when the dominator in (17) is minimal. Thus, for laminar flow \( (n=1/2) \) the greatest error is \( g_1 / 2 \), and for turbulent flow \( (n=1/5) \) it is \( g_1 / 5 \). For laminar flow the coefficient \( g_1 \) is practically constant for \( Pr > 0.5 \) and according to Table 3 equals \( g_1 = 0.62 \), the maximum error in this case is \( \approx 30\% \). For turbulent flow at \( Pr = 1 \), the coefficient \( g_1 = 0.2 \) (Table 3, row 8) and it decreases when \( Pr \) increases. Therefore, in this range of Prandtl number, the maximum error is \( \approx 4\% \). However, for \( Pr = 0.01 \), the coefficient \( g_1 = 0.5 \), and hence, the error is \( \approx 10\% \).

Thus, in this problem for laminar flow, the error may be moderate, while for turbulent flow, when \( Pr > 1 \), the use of the common approach does not lead to significant errors. These estimates are in agreement with corresponding conjugate problem solutions. For laminar flow, conjugate solution gives the maximum error from 20 to 25%. For turbulent flow, the maximum error according to conjugate solution is about 7%. These results are obtained in conjugate solution considering the plate as a thin with linear temperature distribution across the thickness which is the same assumption as that using in derivation of formula (3) (details in [24]).

Note that the errors in this problem are moderate in laminar and small in turbulent flows because the temperature head increases in flow direction on both sides of the plate.

Example 10 Heat transfer from a symmetrically streamlined thermally thin plate.
In this example, we estimate the error for the problem conjugate solution of which is obtained in example 1. The averaged conduction equation (4) and its common solution under boundary condition of the third kind with isothermal heat transfer coefficient are

\[
\frac{d^2 \theta_w}{dx^2} - 2Bi_{L} \theta_w = 0,
\quad \theta_w = \frac{T_w - T_\infty}{T_L - T_\infty} = \frac{ch\left(\sqrt{2Bi_{L}} \frac{\xi}{L}\right)}{ch\left(\sqrt{2Bi_{L}} \frac{\xi}{L}\right)},
\]

\[
\tilde{h}_L = \frac{L^2}{\lambda \Delta}
\]  

(18)

Here, a bar at the top of the parameters indicates the averaged values, like, for example, the averaged along the plate heat transfer coefficient \( \tilde{h}_L = 2h_{L} \). The first equation (18) is obtained after substituting the heat flux \( q_w = \tilde{h}_L \theta_w \) for heat fluxes in equation (4) since in common approach usually the average isothermal heat transfer coefficient is used (see introduction). The solution of equation (18) is obtained applying the hyperbolic cosine function because the second derivative of the hyperbolic cosine is proportional to this function itself as it requires a solution of differential equation (18).

Calculating the second term of the last expression (10) using solution (18) yields an error caused by this solution due to neglecting interface temperature distribution. After differentiating these second equation (18), one gets error and then, putting \( \sigma_{\max} = 1 \), obtains the maximal value of error

\[
\sigma = g_1 \frac{d\theta_w}{dx} = g_1 \sqrt{2Bi_{L}} \frac{x}{L} \tilde{h}_L \left(\sqrt{2Bi_{L}} \frac{\xi}{L}\right),
\quad \sigma_{\max} = g_1 \sqrt{2Bi_{L}} = g_1 \sqrt{\frac{2Nu_{L}L}{\lambda_\Delta}}
\]

(19)

The last equation (19) follows from a previous after using Nusselt number \( Nu_{L} = hL/\lambda \) instead of Biot number. Calculation yields: \( Re = 5 \cdot 10^4 \) (laminar flow), \( Nu_{L} = 2Nu_{L} = 132, Bi_{L} = 1.32, g_1 = 0.62 \) and \( \sigma_{\max} \approx 1 \). Hence, the solution of this problem with the boundary condition of the third kind is unacceptable, and conjugate solution is required.

The conjugate solution of this problem is given in Table 1. Calculation shows that the temperature head distribution obtained by common simple solution (18) differs significantly from conjugate results with the largest deviation at the leading edge of 36% giving 0.379 instead of 0.278. This problem as well as previous problem in the case of laminar flow (with an error of about 30%) are examples with increasing along the plate temperature head when the conjugate solution is required.

**Example 11** Heat transfer from continuous plate (strip) of polymer

A strip of polymer at temperature \( T_0 \) is extruded from a die and passed at velocity \( U_w \) through a bath with cold water \( (Pr = 6.1) \) at temperature \( T_\infty \). The solution of this problem with boundary condition of the third kind is presented in paper [33] and in the book [24]. The along the strip dimensionless temperature head distribution is presented in the form \( \Theta_w(\tilde{x}) \), with variables \( \theta_w = (T_w - T_\infty) / (T_L - T_\infty) \) and \( \tilde{x} = x\alpha_w / \Delta U_w \), where \( \alpha_w \) is the thermal diffusivity. It is shown that in this case, the dimensionless temperature head depends on the ratio \( (c_r \rho \lambda)w / (c_r \rho \lambda) \) which is 8.51 in considered example. The derivative \( d\theta_w / dx \) required for error estimation is found by numerical differentiation of the corresponding curve \( \theta_w(\tilde{x}) \). Estimation of the second term of the last equation (10) shows that the error grows as the distance from the die increases and finally reaches the maximal value \( \sigma_{\max} \approx 2.6 \). It is evident that the real results may be obtained only by solving the conjugate problem. The reason of that is a decreasing temperature head resulting in highly overestimating temperature of cooling strip [33], [24].

Presented here examples demonstrated that the error estimation by calculating the second term of series (2) for heat flux or of the second term of the last equation (10) for nonisothermicity coefficient helps to understand whether the conjugate solution for particular problem is required or the common simple approach is satisfactorily accurate.
Conclusion

Should Any Heat Transfer Problem Be Considered as a Conjugate?

The theory and analysis of examples show that the level of conjugation (the difference between conjugate and traditional solutions) of a particular convective heat transfer problem depends on many factors basic of which are:

- Variation of the temperature head in flow direction or in time. The decreasing temperature head affects the heat transfer characteristics much more strongly than an increasing temperature head.
- Relation between thermal resistances of the body and fluid at isothermal conditions characterized by Biot number. The level of conjugation is greater for the case of comparable resistances and usually small when one of thermal resistances is negligible.
- Parameters and conditions determining the coefficients in the basic expression (2) for heat flux on nonisothermal surface: (i) flow regime- the effect of conjugation in laminar flows is greater than that in turbulent flows, (ii) state of heat transfer- the effect of conjugation at unsteady heat transfer is higher than that for steady-state temperature regime, (iii) Prandtl number- the higher is the Prandtl number, the smaller the effect of conjugation, (iv) Reynolds number- the effect of conjugation decreases with growing the Reynolds number, (v) type of coolant- for some non-Newtonian fluids, the effect of conjugation is greater (for \( n > 1 \)) but for others is smaller (for \( n < 1 \)) than for Newtonian liquids, (vi) shape of the surface- for example, increasing surface curvature leads to increasing the conjugation effect, (vii) type of boundary layer- for instance, a conjugation effect in a flow on continuously moving sheet is greater than that in flow past fixed plate.
- Distribution of the pressure gradient. The effect of conjugation is usually higher in the flows with unfavorable gradients.
- A scheme of heat transfer. For example, the effect of conjugation is greater for the heat transfer between countercurrent flowing fluids than that for concurrent flowing fluids under the same conditions.

In Table 4 below, the factors affecting the conjugation effect are listed and arranged so that next to the right issue, each represents a subject with lower effect of conjugation. For instance, because turbulent flow is located to the right of the laminar flow, this means that the conjugation effect in the problems with turbulent flows is less than that in corresponding problems with laminar flows.

<table>
<thead>
<tr>
<th>Decreasing temperature heads</th>
<th>Increasing temperature heads</th>
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<tr>
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It is obvious that in reality, the choice of the method for solution largely depends on the aim of a particular heat transfer problem and on the desired accuracy of results. Therefore, such qualitative considerations can be used for preliminary approximate estimation, whereas the exact information of the conjugate effect can be obtained only by
solving a particular conjugate problem. Nevertheless, the investigation results make it possible to formulate two general conclusions regarding question formulated in heading:

1. Convective heat transfer problems containing a temperature head decreasing in flow direction or in time should be, as a rule, considered as a conjugate because in this case, the effect of conjugation is usually significant.

2. For the turbulent flow of the fluids with high (say higher than 100) Prandtl numbers, the convective heat transfer problems may be solved using a traditional approach with boundary condition of the third kind, because for such fluids, the effect of conjugation is negligible.

For other cases, the error arising by using the traditional approach may be estimated by computing the second term \( g_1 \frac{x}{\omega} \frac{d\phi^*}{dx} \) of equation (10) for nonisothermisity coefficient applying the known traditional solution. The value of this term helps to understand whether the conjugate solution is required. Examples of such estimations are given in the part 5 of this text.

REFERENCES


