Hygrothermoelastic Nonlinear Flexural Response of Graphite Epoxy Laminated Composite Plates with Uncertain System Properties

Rajesh Kumar*1, Million Merid Afessa1
School of Mechanical Engineering, Jimma Institute of Technology, Jimma University, Jimma, Ethiopia
*rajeshtripathi63@gmail.com; millionmerid208@gmail.com

Abstract
This paper presents Hygrothermoelastic nonlinear flexural response of graphite epoxy composite plates with uncertain system properties. Lamina material properties, geometric properties, coefficients of thermal expansion, coefficients of hygroscopic expansion and lateral load are modeled as basic random variables using micromechanical model. A higher order shear deformation theory in the von-Karman sense is used to model the system behavior of the laminated plate. A direct iterative based C0 nonlinear finite element method in conjunction with the first order perturbation technique developed earlier is extended for hygrothermal problem to obtain the second order response statistics, i.e., mean and coefficient of variations of nonlinear transverse central deflection of the plate. Typical numerical results are obtained for various combinations of geometric parameters, uniform lateral pressures, staking sequences, volume fractions, aspect ratios, plate thickness ratios, boundary conditions under environmental conditions. The results obtained have been compared with those available in the literature and an independent Monte Carlo simulation.

Keywords
Micromechanical Model; Nonlinear Bending Response; Stochastic Finite Element Method; Uncertain System properties

I. Introduction
Composite laminated structures are being increasingly used in aeronautical and aerospace construction due to gaining wide popularity as lightweight components, ability to tailor structural properties through appropriate lamination scheme for achieving high strength and stiffness to weight ratio. These plates are often combination of transverse mechanical and hygrothermal loading. Due to low shear modulus compared to in-plane Young’s modulus, transverse shear deformation even more pronounced in composite laminates. The capability to predict the structural response and enable a better understanding and characterization of the actual behavior of laminated composite plates when subjected to combined load is of prime interest to structural analysis. In fact, many structures are subjected to high load levels that may result in nonlinear load-deflection relationships due to large deformations of the plates. One of the important problems deserving special attention for accurate prediction of structural response in sensitive application is the study of their nonlinear response to large deflections by assuming random system properties as independent random variables.

During typical operating conditions structures are constantly being subjected to random load like engine noise, shocks waves, turbulence, gusts, track inputs, thermal loads, winds and acoustic loads. Therefore, the external loading is also random in nature. Some of these structures are often subjected to severe loading that result in large response and consequently demand the investigation. For reliability of design, accurate prediction of system behaviour of the laminated composite structures in the presence of randomness in the system properties is needed. A considerable volume of literature is available on the static response of geometrically linear and nonlinear...
composite laminated plates under various thermal and mechanical loads or combination of two. (See examples [1-9]). All based on the assumptions of the complete determinacy of structural parameters. In the deterministic analysis of structures, the variations in the system parameters are neglected and mean value of system parameters are used in the analysis. Due to the dependency of large numbers of parameters in complex production and fabrication processes of laminated composite plate. The system properties can be random in natures resulting in uncertainty in the response of the plate. Therefore to well define the original problems and enable a better understanding and characterization of the actual behaviour of the laminated composite structures, it is obviously of prime important that the inherent randomness in system parameters be incorporated in the analysis. Relatively little effort has been made in the past by the researchers and investigators on the prediction of the thermo-mechanical bending response of the structures made of laminated composites plates having random system properties. Based on higher order theory, Naveenth Raj et al. [10] have evaluated the linear static response statistics of graphite-epoxy composite laminates with randomness in material properties for different boundary conditions, thickness ratios, aspect ratios and fibre orientations to deterministic loading by using combination of finite element analysis and Monte Carlo simulation (MCS). Salim et al. [11] also examined the effect of randomness in material properties (like elastic modulus Poisson’s ratios etc.) on the response statistics of a composite plate subjected to static loading using classical plate theory (CLT) in conjunction with first order perturbation techniques (FOPT). Onkar and Yadav [12] have investigated the non-linear response statistics of composite laminated flat panel with random material properties subjected to transverse random loading based on CLT in conjunction with FOPT. Yang et al. [13] have investigated the stochastic bending response of moderately thick compositionally graded plates with random system properties under lateral load and uniform temperature change. They have utilized a first order perturbation technique to obtain the response statistics, while basic formulation of the problem has been developed based on Reddy’s higher order shear deformation theory (HSDT). Zongeen and Suhaun [14] presented a method to estimate the standard deviation of eigenvalue and eigenvector of random multiple degree of freedom system. Zhang et al. [15] have applied the stochastic perturbation method to vector-valued and matrix-valued function for the response and reliability of uncertain structures. Liu et al. [16] formulated the probabilistic finite element method (PFEM) for linear and nonlinear continua with homogeneous random fields of a one dimensional elastic plastic wave propagation problems and a two dimensional plane-stress beam bending problem. Zhang and Ellingwood [17] examined the effect of random material field characteristics on the instability of a simply supported beam on elastic foundation and a frame using perturbation technique. Noh [18] framed stochastic finite element analysis to investigate the effect of multiple uncertain material properties on the response variability of in-plane and plate structures with multiple uncertain material parameters. Keeping in mind above aspect, in the present work, an HSDT proposed by Lal et al. [19] and Singh et al. [20] is extended to random environments. They presented $C^0$ linear and nonlinear finite element method (FEM) in conjunction with a FOPT to obtain the second order response statistics of bending deflection of laminated composite plate supported with and without elastic foundation. They included the transverse shear effects in the system equation using HSDT. In order to incorporate the uncertainties of the physical properties of laminated composite structures, a stochastic finite element based second moment was developed by Park et al. [21]. Pandit et al. [22 and 23] presented the improved higher order plate model to study the response statistics of a soft core sandwich plates. A computationally efficient $C^0$ stochastic finite element method (SFEM) based on mean centered FOPT has been proposed to obtain the second order statistics of deflection of sandwich plate under transverse loading. Lal et al. [24,25] studied the effect of random system properties on bending response of thermo-mechanically loaded laminated composite plates and stochastic nonlinear bending of thermo-mechanically loaded (consists of a lateral pressure and thermal loading) laminated composite plates for macro mechanical model in the presence of small random variation in the system variables taking into account the transverse shear strain using the HSDT with von-Karman nonlinear strain displacement relations. Shen et al. [26] studied the hygrothermal effects on the nonlinear bending of shear deformable laminated plates using deterministic finite element method and micromechanical model. Upadhyay et al. [27] investigated the nonlinear flexural response of laminated composite plates under hygro-thermo-mechanical loading using deterministic finite element method and micromechanical model. R Kumar et al. [37,38,39] investigated the linear and nonlinear flexural response of laminated composite plates with random material properties and plates restin
on a Nonlinear Elastic Foundation with Uncertain System Properties under Lateral Pressure and Hygrothermal Loading.

To the best of the authors’ knowledge, there is no literature covering the second order response statistics of geometrically nonlinear laminated composite plates, subjected to combined lateral pressure and hygrothermal loading involving randomness in system properties for micromechanical model using computationally efficient C0 nonlinear finite element method in conjunction with mean centred first order perturbation technique (FOPT). This is the problems studied in the present paper.

In the present study, the stochastic nonlinear bending of hygrothermo-mechanically loaded (consists of a lateral pressure and hygrothermal loading) laminated composite plates in the presence of small random variation in the system variables taking into account the transverse shear strain using the HSDT with von-Karman nonlinear strain displacement relations is studied. A direct iterative based C0 nonlinear FEM in conjunction with the mean centered FOPT as developed by the authors is extended and employed to determine the second-order-statistics (mean and standard deviation) of nonlinear transverse central deflection of laminated composite plates subjected to uniform constant temperature and moisture(U.T). The numerical illustrations concern the stochastic nonlinear bending response of laminated composite plate subjected to uniform temperature and moisture distribution over plate surface and through the plate thickness are obtained for various combinations of geometric parameters, uniform lateral pressures, staking sequences, volume fraction, aspect ratio, plate thickness ratio, boundary conditions under environmental conditions. It is observed that small amount of random system properties variations of the composite plate in the presence of temperature and moisture(hygrothermal) significantly affect the nonlinear transverse central deflection.

The proposed probabilistic procedure would be valid for system properties with small random coefficient of variations compared to their mean value which is usually satisfied by most engineering applications.

II. Formulations

Consider geometry of laminated composite rectangular plate of length a, width b, and thickness h, which consists of N−plies located in three dimensional Cartesian coordinate system (X, Y, Z) where X- and Y-plane passes through the middle of the plate thickness with its origin placed at the corner of the plate as shown in Fig. 1. Let \( (\bar{u}, \bar{v}, \bar{w}) \) be the displacements parallel to the (X, Y, Z) axes, respectively. The thickness coordinate Z of the top and bottom surfaces of any \( k\)th layer are denoted by \( Z_{(k-1)} \) and \( Z_k \) respectively. The fiber of the \( k\)th layer is oriented with fiber angle \( \theta_k \) to the X-axis. The plate is assumed to be subjected to uniformly distribute transverse static load is defined as \( q(x, y) = q_o \).

2.1 Displacement field model

In the present study, the assumed displacement field is based on the Reddy’s higher order shear deformation theory [28], which requires C1 continuous element approximation. In order to avoid the usual difficulties associated with these elements the displacement model has been slightly modified to make the suitability of C0 continuous element [29]. In modified form, the derivatives of out-of-plane displacement are themselves considered as separate degree of freedom (DOFs).

Thus five DOFs with C1 continuity are transformed into seven DOFs with C0 due to conformity with the HSDT. In this process the artificial constraints are imposed which should be enforced variationally through a penalty approach. However the literature [29] demonstrates that without enforcing these constraints the accurate results using C0 can be obtained. The modified displacement field along the X, Y, and Z directions for an arbitrary composite laminated plate is now written as

\[
\begin{align*}
\bar{u} &= u + f_1(z)\phi_x + f_2(z)\phi; \\
\bar{v} &= v + f'_1(z)\phi_y + f'_2(z)\phi; \\
\bar{w} &= w_x.
\end{align*}
\]
where \( u, v, \text{ and } w \) denote the displacements of a point along the \((X, Y, Z)\) coordinates axes: \( u, v \) and \( w \) are corresponding displacements of a point on the mid plane, \( \phi_x = w_x, \text{ and } \phi_y = w_y \), and \( \psi_x, \psi_y \) are the rotations of normal to the mid plane about the y-axis and x-axis respectively. The functions \( f_1(z) \) and \( f_2(z) \) can be written as

\[
f_1(z) = C_1 z - C_2 z^3; \quad f_2(z) = -C_4 z^2 \text{ with } C_1 = 1, \quad C_2 = C_4 = 4h^2/3.
\]

The displacement vector for the modified \( C^0 \) continuous model is denoted as

\[
\{ \Lambda \} = [u \quad v \quad w \quad \phi_x \quad \phi_y \quad \psi_x \quad \psi_y]^T,
\]

where, comma (,) denotes partial differential.

### 2.2 Strain Displacement Relations

For the structures considered here, the relevant strain vector consisting of strains in terms of mid-plane deformation, rotation of normal and higher order terms associated with the displacement for kth layer are as

\[
\{ \varepsilon \} = \{ \varepsilon_i \} + \{ \varepsilon_{na} \} - \{ \varepsilon_{HT} \}
\]

where \( \{ \varepsilon_i \}, \{ \varepsilon_{na} \} \) and \( \{ \varepsilon_{HT} \} \) are the linear and nonlinear strain vectors, hygrothermal strain vector, respectively.

Using Eq. (3) the linear strain vector can be obtained using linear strain displacement relations \([29, 30]\), which can be written as

\[
\{ \varepsilon_i \} = \begin{bmatrix} \varepsilon_{i,x} \\ \varepsilon_{i,y} \end{bmatrix} + \begin{bmatrix} z \varepsilon_{i,x} \\ z^2 \varepsilon_{i,x} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} z^2 \varepsilon_{i,x} \\ 0 \end{bmatrix}.
\]

where,

\[
\begin{bmatrix} \varepsilon_{i,x} \\ \varepsilon_{i,y} \end{bmatrix} = \begin{bmatrix} \varepsilon_{ix} \\ \varepsilon_{iy} \end{bmatrix}, \quad \begin{bmatrix} \varepsilon_{i,x} \\ \varepsilon_{i,y} \end{bmatrix} = \begin{bmatrix} \psi_{x,x} \\ \psi_{x,y} \end{bmatrix}, \quad \begin{bmatrix} \varepsilon_{i,x} \\ \varepsilon_{i,y} \end{bmatrix} = \begin{bmatrix} \psi_{x,y} + \psi_{y,x} \\ \psi_{x,y} + \psi_{y,x} \end{bmatrix},
\]

\[
\{ \varepsilon^* \} = -C_2 \begin{bmatrix} \psi_{x,x} \\ \psi_{x,y} \end{bmatrix} - C_4 \begin{bmatrix} \theta_{x,x} \\ \theta_{x,y} \end{bmatrix},
\]

\[
\{ \varepsilon_{na} \} = C_1 \begin{bmatrix} \psi_{x,x} \\ \psi_{x,y} \end{bmatrix} + \begin{bmatrix} w_x \\ w_y \end{bmatrix}, \quad \{ \varepsilon^*_na \} = -C_2 \begin{bmatrix} \psi_{x,x} \\ \psi_{x,y} \end{bmatrix} - C_4 \begin{bmatrix} w_x \\ w_y \end{bmatrix},
\]

Assuming that the strains are much smaller than the rotations (in the von-Karman sense), one can obtain nonlinear strain vector \( \{ \varepsilon_{na} \} \) of the Eq. (4) as \([30]\)

\[
\{ \varepsilon_{na} \} = \frac{1}{2} [A_{na}] \{ \phi \}
\]

Where

\[
[A_{na}] = \frac{1}{2} \begin{bmatrix} w_{x,x} & 0 \\ 0 & w_{x,y} \\ w_{x,y} & w_{y,y} \end{bmatrix} \text{ and } \{ \phi \} = \begin{bmatrix} w_{x,x} \\ w_{x,y} \\ w_{y,y} \end{bmatrix},
\]

\[
\{ \phi \} = \begin{bmatrix} w_{x,x} \\ w_{x,y} \\ w_{y,y} \end{bmatrix}.
\]
The hygrothermal strain vector \( \{ \varepsilon_{HT} \} \) is represented as [36]

\[
\{ \varepsilon_{HT} \} = \{ \varepsilon_x \} = \Delta T \{ \alpha_{x} \} + \Delta C \{ \beta_{x} \}
\]

(8)

\( \alpha_{x}, \alpha_{y} \) and \( \alpha_{z} \) are coefficients of thermal expansion and \( \beta_{x}, \beta_{y} \) and \( \beta_{z} \) are coefficients of hygroscopic expansion along the \( x, y, z \) direction respectively which can be obtained from the thermal coefficients in the longitudinal \( (\alpha) \) and transverse \( (\alpha) \) directions of the fibers using transformation matrix and \( \Delta T \) is the change in temperature and the change in moisture in percentage in the plate subjected with uniform moisture \( (\Delta C=C_0 \text{ in percentage}) \) and temperature \( (\Delta T=T_0 \text{ rise (U.T)} ) \).

### 2.3 Stress–strain relation

The constitutive law of thermo-elasticity for the materials under considerations relates the stresses with strains in a plane stress state for the \( k \)th lamina oriented as an arbitrary angle with respect to reference axis for the orthotropic layers is given by [30] \( \sigma_i = [\overline{Q}]_{ij} \varepsilon_j \)

Or

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z
\end{bmatrix} =
\begin{bmatrix}
\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\
\overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\
\overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z
\end{bmatrix} + \overline{Q}_{44} \{ \varepsilon_r \} + \overline{Q}_{45} \{ \varepsilon_m \} + \overline{Q}_{55} \{ \varepsilon_t \}
\]

(9)

where, \( \{ \overline{Q} \} \), \( \{ \sigma \} \) and \( \{ \varepsilon \} \) are transformed stiffness matrix, stress and strain vectors of the \( k \)th lamina, respectively.

### 2.4 Strain energy of the plate

The strain energy \( (\Pi) \) of the laminated composite plate is given by

\[
\Pi = \frac{1}{2} \int_A \{ \varepsilon \}^T [\sigma] \{ \varepsilon \} dA
\]

(10)

Using Eqs. (3)-(8), the strain energy as given above can be written as

\[
\Pi = \Pi_l + \Pi_u
\]

(11)

Where, \( \Pi_l \) and \( \Pi_u \) are the linear and the nonlinear strain energy respectively which are expressed as

\[
\Pi_l = \frac{1}{2} \int_A \{ \varepsilon \}^T [\overline{Q}] \{ \varepsilon \} dA
\]

(12a)

\[
\begin{align*}
\Pi_u &= \frac{1}{2} \int_A \{ \varepsilon_u \}^T [\overline{Q}] \{ \varepsilon_u \} dA + \frac{1}{2} \int_A \{ \varepsilon_r \}^T [\overline{Q}] \{ \varepsilon_r \} dA \\
&\quad + \frac{1}{2} \int_A \{ \varepsilon_m \}^T [\overline{Q}] \{ \varepsilon_m \} dA
\end{align*}
\]

(12b)
1) Linear strain energy of the plate
Using linear strain displacement relations [20], the linear elastic strain energy as given in Eq. (12a) can be expressed as

$$\Pi_l = \frac{1}{2} \int_A \{\varepsilon\}^T \left[\overline{Q}\right] \{\varepsilon\} \, dA$$

(13)

Where, \(\{\varepsilon\}\) is the linear strain vector at the reference plane, i.e., \(z=0\) and \([D]\) is the laminate stiffness matrix.

2) Nonlinear strain energy of the plate
Using nonlinear strain displacement relations in the von Karman sense [30,35], the nonlinear energy as given in Eq. (12b) can be expressed as

$$\Pi_n = \frac{1}{2} \int_A \{\varepsilon\}^T \left[D_3\right] \{\varepsilon\} \, dA + \frac{1}{2} \int_A \{\phi\}^T \left[D_4\right] \{\phi\} \, dA + \frac{1}{2} \int_A \{\phi\}^T \left[D_5\right] \{\phi\} \, dA$$

(14)

Where, \([D_3]\), \([D_4]\) and \([D_5]\) are the laminate stiffness matrices as given in appendix and \([A]\) and \([\phi]\) are defined in appendix.

2.5 Potential energy due to hygrothermal stresses
The potential energy (\(\Pi_2\)) storage by hygrothermal (combined temperature and moisture) load is written as

$$\Pi_2 = \frac{1}{2} \int A \left( N_x(w_x)^2 + N_y(w_y)^2 + 2N_{xy}(w_x)(w_y) \right) \, dA$$

(15)

where, \(N_x, N_y\) and \(N_{xy}\) are pre-buckling hygrothermal stresses.

2.6 External work done
The potential energy due to distributed transverse static load \(q(x, y)\) can be expressed as

$$\Pi_3 = W_{ext} = -W_q = \int_A q(x, y) \, w \, dA$$

(16)

where, \(q(x, y)\) is the intensity of distributed transverse static load which is defined as

$$q(x, y) = \frac{QEh}{b^3}$$

here \(Q\) is represented as uniform lateral load.

2.7 Finite element model
In the present study a \(C^0\) nine-noded isoparametric finite element with 7 DOFs per node is employed. The domain is discretized into a set of finite elements. Over each of the element, the displacement vector and the element geometry are expressed as

$$\{\Lambda\} = \sum_{i=1}^{NN} \phi_i \{\Lambda_i\}; \quad x = \sum_{i=1}^{NN} \phi_i x_i; \text{ and } y = \sum_{i=1}^{NN} \phi_i y_i$$

(17)
where, $\varphi_i$ is the interpolation (shape function) function for the $i$th node, $\{\Lambda\}_i$ is the vector of unknown displacements for the $i$th node, $NN$ is the number of nodes per element and $x_i$ and $y_i$ are Cartesian coordinate of the $i$th node.

1) Strain energy of the laminated plate

The linear and nonlinear functional are computed for each element and then summed over all the elements in the domain to get the total functional. Following this, and using Eq. (17), Eq. (10) can be written as

$$\Pi = \sum_{e=1}^{NE}(\Pi_{\text{el}}^{(e)} + \Pi_{\text{nl}}^{(e)})$$

where,

$$\Pi_{\text{el}}^{(e)} = \frac{1}{2} \int_{A_e} \left\{ [\Lambda^{(e)}, K_{2e}] \right\}^{T} \{\Lambda^{(e)}\} dA + \frac{1}{2} \int_{A_e} \left\{ [\Lambda^{(e)}, K_{3e}] \right\}^{T} \{\Lambda^{(e)}\} dA$$

$$\Pi_{\text{nl}}^{(e)} = \left\{ [\Lambda^{(e)}, K_{nl}] \right\}^{T} \{\Lambda^{(e)}\}$$

Here, $[K_{\text{el}}](e), [K_{\text{nl}}](e)$ and $[K_{\text{nl}}](e)$ are the elemental nonlinear stiffness matrices. $[K_{\text{el}}](e)$ is the linear stiffness matrix and $[\Lambda](e)$ is the elemental nodal displacement vector.

Following the assembly procedure, Eq. (18) can be further written as

$$\Pi_1 = \frac{1}{2} \{q\}^{T} \left[ K_{b} + K_{s} \{q\} \right] \{q\} - \{q\}^{T} \left[ F^{HT} \right]$$

With $[K_{b}] = [K_{s}] + [K_{s}] [K_{nl}] + [K_{s}]$,

where global bending stiffness matrix $[K_{b}]$, shear stiffness matrix $[K_{s}]$, global nonlinear stiffness matrix $[K_{nl}]$, global displacement vector $[q]$ and hygrothermal load vector $[F^{HT}]$ are defined in the appendix.

2) Hygrothermal buckling analysis

Using finite element model (Eq. (20)), Eq. (18) can also be written as

$$\Pi_2 = \sum_{e=1}^{NE} \Pi_{\text{el}}^{(e)} = \frac{1}{2} \lambda \{q\}^{T} \left[ K_{s} \right] \{q\}$$

Were, $\lambda$ and $[K_{s}]$ are defined as the hygrothermal buckling load parameters and the global geometric stiffness matrix, respectively.

3) Work done due to external transverse load

Using finite element model (Eq. (20)), Equation (19) may be written as

$$\Pi_3 = \sum_{e=1}^{NE} \Pi_{\text{nl}}^{(e)}$$

where

$$\Pi_{\text{nl}}^{(e)} = -\int_{A_e} \left\{ [\Lambda]^{(e)} \right\}^{T} \left\{ P_{M} \right\} \{\Lambda\} dA$$

$$= -\{q\}^{T} \left\{ P_{M} \right\} \{\Lambda\}$$

$$= -\{q\}^{T} \left\{ P_{M} \right\} \{\Lambda\}$$
with \( \{ P_m \}^{(r)} = (0 \ 0 \ q \ 0 \ 0 \ 0)^T \)\

Adopting Gauss quadrature integration numerical rule, the element stiffness and geometric stiffness matrices, load vectors, respectively can be obtained by transforming expression in \( x, y \) coordinate system to natural coordinate system \( \xi, \eta \).

### III. Governing equation

The governing equation for the nonlinear static analysis can be derived using Variational principle, which is generalization of the principle of virtual displacement [30]. For the bending analysis, the minimization of first variation of total potential energy \( \Pi (\Pi_1 + \Pi_2 + \Pi_3) \) with respect to displacement vector is given by

\[
\delta (\Pi_1 + \Pi_2 + \Pi_3) = 0
\]

by substituting Eqs. and simplification gives us.

\[
[K_s] [W] = [F],
\]

with \( [K_s] = [K_l + K_{nl} \{ q \}] \) and \( [F] = \{ P_m \} + \{ P_{HT} \} \)

Where, \([K_l], [K_{nl}], [P_{M}]\)and\([P_{HT}]\) are global linear, nonlinear stiffness matrix, global mechanical and hygrothermal force vector respectively defined in appendix. The stiffness matrix \([K_s]\), displacement vector \([W]\) and force vector \([F]\) is random in nature, being dependent on the system properties. Therefore the eigenvalues and eigenvectors also become random. In deterministic environment, the solution of Eq. (25) can be obtained using iterative, incremental methods etc. However in random environment, it is not possible to obtain the solution using above mentioned numerical methods. Further analysis is required to obtain the complete solution of solution of Eq. (25).

For this purpose novel probabilistic procedure as developed by the authors [20] is extended for this problem in the present work. The direct iterative method combined with nonlinear finite element method, i.e., direct iterative based nonlinear finite element method in conjunction with mean centered FOPT (DISFEM) with a reasonable accuracy to obtain the second order statistics of nonlinear static response.

### 3.1 Solution approach — a DISFEM for nonlinear static bending problem

The random governing equation as given in Eq. (30) is solved by employing a DISFEM, assuming that the random changes in the transverse displacements do not affect much the nonlinear stiffness matrices with the following steps:

(i) The stiffness matrix \([K_s]\) is obtained in the first step neglecting all nonlinear terms, yielding the linear stiffness matrix. Using the linear stiffness matrix and the displacement vector, the random governing equation is broken into the zeroth order and the first order equations using the perturbation technique. Then the displacement vector \([W]\) is obtained from any standard deterministic method using the zeroth order equation. The first order perturbation technique as presented in the next Section 4.2 is then employed to obtain the standard deviation of the displacement response using the first order equation.

(ii) The displacement vector is normalized. For a specified maximum deflection \( C \) at the centre of the plate, the displacement vector is scaled up by \( C \) times, so that the resultant vector will have a displacement \( C \) at the maximum deflection point.

(iii) Using the scaled-up normalized displacement vector, the nonlinear terms in stiffness matrix \([K_s]\) can be obtained. The problem may now be treated as a linear random static problem with the updated nonlinear stiffness matrix. The random linear static problem can again be broken up as stated in step (i) into the zeroth and the first order equations. The zeroth order can be used to obtain the nonlinear displacement vector \([W_{nl}]\) and the random first order equations can be used to obtain standard deviation of the nonlinear displacement vector using the first order perturbation technique as presented in Section 4.2.
(iv) Steps (ii) and (iii) are repeated to obtain the converged nonlinear response vector \( \{ W_{nl} \} \) to a prescribed accuracy (say \( \approx 10^{-3} \)).

(v) Steps (i) – (iv) are repeated for various values of C.

### 3.2 Solution approach: perturbation technique

The governing equation (25) can be written in the most general form as:

\[
\begin{bmatrix} K^R_s \\ W^R \end{bmatrix} \{ F^R \} = \{ R \}
\]  

(26)

Where, \( \begin{bmatrix} K^R_s \\ W^R \end{bmatrix} \) and \( \{ F^R \} \) are represented as the random stiffness matrix, the random response vector and the random force vector respectively and superscript ‘\( R \)’ denotes random.

Any random variable can be expressed as the sum of its mean and the zero mean random variable which is expressed as:

\[
\text{random variable}(RV) = \text{mean}(R\text{V}) + \text{zero-mean random variable}(R\text{V}')
\]

The operating random variables in the present case are defined as [31, 32]

\[
\begin{align*}
& b^R = b^d + b^r; \quad K^R_s = K^d_s + K^r_s; \quad W^R = W^d + W^r; \quad F^R = F^d + F^r \\
\end{align*}
\]

(27)

We can express the above relations in the form:

\[
\begin{align*}
& b^R = b^d + \varepsilon b^r; \quad K^R_s = K^d_s + \varepsilon K^r_s; \quad W^R = W^d + \varepsilon W^r; \quad F^R = F^d + \varepsilon F^r
\end{align*}
\]

(28)

Where, \( \varepsilon \) is a scaling parameter, and is small in magnitude. The superscripts ‘\( d \)’ and ‘\( r \)’ denote the mean and zero mean random part.

We consider a class of problems where the zero-mean random variation is very small as compared to its mean part. Using the Taylor series expansion and neglecting the second and higher-order terms since the first order approximation is sufficient to yield results with desired accuracy having low variability as is the case in most of the sensitive application [31, 33 and 34]. Substituting Eq. (28) in Eq. (26) we get:

\[
\begin{bmatrix} K^d_s + \varepsilon K^r_s \\ W^d + \varepsilon W^r \end{bmatrix} \{ F^d + \varepsilon F^r \} = \{ R \}
\]

(29)

Equating the terms of same order, we obtain the zeroth order perturbation equation and first order perturbation equation as follows [34].

The zeroth order perturbation equation(\( \varepsilon^0 \)):

\[
\begin{bmatrix} K^d_s \\ W^d \end{bmatrix} \{ F^d \} = \{ R \}
\]

(30)

The first order perturbation equation(\( \varepsilon^1 \)):

\[
\begin{align*}
& \begin{bmatrix} K^d_s \\ W^r \end{bmatrix} \{ W^d \} + \begin{bmatrix} K^r_s \\ W^d \end{bmatrix} \{ F^d \} = \{ F^r \}
\end{align*}
\]

(31)

Obviously, the zeroth order Eq. (35) is the deterministic and gives the mean response. The first order Eq. (31) on other hand represents its random counterpart and solution of this equation provides the statistics of the nonlinear bending response, which can be solved using the probabilistic methods like perturbation technique, Monte Carlo simulation, Newman’s expansion technique [31, 33 and 34].

Using Taylor’s series expansion the system matrix, the displacement vector and forced vector can be expressed as [34]:
Substituting Eq. (32) in Eq. (31) and equating the coefficients of \( b_i^r \). For each \( l \), we get:

\[
\begin{bmatrix}
K_d^l \\
W_d^l \\
F_d^l
\end{bmatrix} \left[ \frac{\partial W_d^l}{\partial b_i^r} \right] = \left[ \frac{\partial F_d^l}{\partial b_i^r} \right], \quad l = 1, 2, \ldots
\] (33)

Using Eq. (33), the total deflection response and its variance can be written as [32]

\[
W = W_d^l + \left[ \frac{\partial W_d^l}{\partial b_i^r} \right] b_i^r \quad \text{and} \quad \text{var}(W) = \left[ \sum_l \frac{\partial W_d^l}{\partial b_i^r} b_i^r \right]^2
\] (34)

Where \( E[\ ] \) and \( \text{var}(\cdot) \) are the expectation and variance respectively. The variance can further be written as [32]

\[
\text{var}(W) = \sum_l \sum_{i,j} \text{diag} \left[ \frac{\partial W_d^l}{\partial b_i^r} \left( \frac{\partial W_d^l}{\partial b_j^r} \right)^T \right] E(b_i^r, b_j^r)
\] (35)

where, \( N \) is the number of variables and \( E(b_i^r, b_j^r) \) is determined from the autocorrelation function of the underlying stochastic field of \( b \), which can be written as

\[
E(b_i^r, b_j^r) = \sigma_s \rho \sigma_s
\] (36)

where, \( \sigma_s \) and \( \rho \) are the standard deviation (SD) of random variables, the correlation coefficient matrix and number of random variables, respectively. Substituting Eq. (42) in Eq. (41), we obtain as:

\[
\text{var}(W) = \left[ \frac{\partial W_d^l}{\partial b_i^r} \right] \sigma_s \rho \sigma_s \left[ \frac{\partial W_d^l}{\partial b_i^r} \right]^T
\] (38)

Eq. (38) expresses the covariance of the deflection in terms of the SD of random variables \( b_i \) (i=1, 2, ..., R) and correlation coefficients. It is evident from Eq. (38) that the response coefficient of variation obtained by using the first perturbation techniques exhibits linear variation with all random variables in material properties, expansion of the thermal coefficients, lamina plate thickness and lateral loading.

IV. Results and discussion

The second-order statistics of nonlinear transverse central deflection is examined for laminated composite plates under lateral pressure and hygrothermal loading subjected to uniform temperature and moisture distribution over plate surface and through the plate thickness are obtained for various geometric parameters, uniform lateral pressures, stacking sequences, volume fraction, aspect ratio, plate thickness ratio, boundary supports under
environmental conditions. A nine noded Lagrange isoparametric element, with 63 DOFs per element for the present HSDT model has been used for discretizing the laminate. Based on convergence study conducted a $(8 \times 8)$ mesh has been used throughout the study.

The mean and coefficient of variation or variance (SD/mean) of the central nonlinear transverse deflection are obtained considering the random material input variables, hygroscopic expansion coefficients, thermal expansion coefficients, geometric parameters and lateral pressure taking combined as well as separately as basic random variables (RVs) as stated earlier. However, the results are only presented taking SD/mean of the system property equal to 0.10 [16] as the nature of the SD variation is linear and passing through the origin. Hence, the presented results would be sufficient to extrapolate the results for other COV value keeping in mind the limitation of FOPT. The basic random system variables such as $E_1$, $E_2$, $G_{12}$, $G_{13}$, $G_{23}$, $\nu_{12}$, $\alpha_1$, $\alpha_2$, $\beta_2$, h and Q are sequenced and defined as

$$b_1 = E_{11}, \quad b_2 = E_{22}, \quad b_3 = G_{12}, \quad b_4 = G_{13}, \quad b_5 = G_{23}, \quad b_6 = \nu_{12}, \quad b_7 = \alpha_1, \quad b_8 = \alpha_2, \quad b_9 = \beta_2, \quad b_{10} = h, \quad b_{11} = Q$$

The following dimensionless nonlinear transverse mean central deflection and uniform lateral pressure has been used in this study.

$$w_{nl} = \frac{Q E_{22} h^4}{b^4}$$

where $w_{nl}$ is dimensionless mean transverse central deflection.

In the present study, various combination of boundary edge support conditions namely, simply supported (S1 and S2), clamped and combination of clamped and simply supported have been used and shown in Fig. 3. These are written as

All edges simply supported (SSSS): S1

$u = v = w = \theta_x = \theta_y = 0$ at $x = 0, a$; $u = v = w = \theta_x = \theta_y = 0$ at $y = 0, b$;

All edges simply supported (SSSS): S2

$u = v = w = \theta_x = \theta_y = 0$, at $x = 0, a$; $u = w = \theta_x = \psi_y = 0$ at $y = 0, b$;

All edges clamped (CCCC):

$u = v = w = \psi_x = \psi_y = \theta_x = \theta_y = 0$, at $x = 0, a$ and $y = 0, b$;

Two opposite edges clamped and other two simply supported (CSCS):

$u = v = w = \psi_x = \psi_y = \theta_x = \theta_y = 0$, at $x = 0$ and $y = 0$;

$u = v = w = \theta_x = \theta_y = 0$, at $x = a$; $u = w = \theta_x = \psi_y = 0$, at $y = b$.

The plate geometry used is characterized by aspect ratios $(a/b) = 1$and 2, side to thickness ratios $(a/h) = 10, 20, 30$ and 40. $T=T_0+\Delta T$; where $T_0$= total temperature, $T_0$ = Initial Temperature, $\Delta T$ = rise in temperature. $C=Co+\Delta C$ where $C$= total moisture concentration, $Co$= Initial moisture concentration, $\Delta C$= rise in moisture concentration. The following material properties are used for computation Shen [26]:

$T_0=25; \quad Co=0; \quad \Delta T=0; \quad \Delta C=0.0$; $Vf=0.5; \quad Ef=230.0 \times 1e9$;

$Gf=9.0 \times 1e9; \quad cfm=0; \quad \rho_c=1.5; \quad \rho_f=1750; \quad cm=0; \quad \alpha_m=45 \times 1e-6; \quad \rho_m=1200; \quad \beta_m=2.68 \times 1e-3; \quad \beta_f=0; \quad Em=(3.51-0.003 \times T-0.142 \times C) \times 1e9;$

$$|$$

$|$$
\[ G_m = E_m \left( \frac{2}{1+n} \right), E_{10} = (V_f E_f + V_m E_m) \]

\[ \alpha_{11} = \frac{V_f \alpha f + V_m \alpha m}{V_f E_f + V_m E_m} \]

\[ \alpha_{22} = \frac{1 + (1 + \nu_f) V_f \alpha f + (1 + \nu_m) V_m \alpha m - \nu_{12} \alpha_{11}}{} \]

\[ \beta_{11} = \frac{V_f \beta f + V_m \beta m}{E_{11}(V_f \rho f c + V_m \rho m) \rho} \]

\[ \beta_{22} = \frac{V_f (1 + \nu_f) c f m \beta f + V_m (1 + \nu_m) \beta m}{(V_f \rho f c + V_m \rho m)} \rho - \nu_{12} \beta_{11} \]

\[ \rho = V_f \rho f + V_m \rho m \]

\[ V_m + V_f = 1 \]

\[ E_{11} = V_f E_f + V_m E_m \]

\[ \frac{1}{E_{22}} = \frac{V_f}{E_f} + \frac{V_m}{E_m} - \frac{V_f V_m}{\sqrt{V_f E_f + V_m E_m}} \]

\[ \frac{1}{G_{12}} = \frac{V_f}{E_f} + \frac{V_m}{G_m} \]

\[ \nu_{12} = V_f \nu f + V_m \nu m \]

E111=0.5×1e-3; E21=0.2×1e-3; G121=0.2×1e-3; G131=0.2×1e-3; G231=0.2×1e-3; \alpha_{11}=0.5×1e-3; \alpha_{21}=0.5×1e-3; \beta_{11}=0.5×1e-3; \beta_{21}=0.5×1e-3; E1(TC)=E10(1+E111(T+C); E2(TC)=E20(1+E21(T+C);

G12(TC)=G120(1+G121(T+C); G13(TC)=G130(1+G131(T+C); G23(TC)=G230(1+G231(T+C); \alpha_{1}(T)=\alpha_{10}(1+\alpha_{11}T);

4.1 Validation study for mean transverse central deflection

The mean transverse central deflection of the of symmetric angle-ply \([\pm45^\circ]_T\) simply supported (S2) laminated composite square plates subjected uniform lateral pressure combined with and without rise in uniform temperature and moisture, volume fraction \((V_f) = 0.6\), plate thickness ratio \(a/h=10\) is shown in Table 1 and compared with [26]. From the table it is clear that the present DISFEM results using HSDT are in good agreement with the available results using higher shear deformation theory of [26].

4.2 Validation study for random nonlinear transverse central deflection

Hence no results are available in reported literatures for structural response of laminated composite plates with system randomness in hygrothermal environments therefore; the outlined DISFEM approach can be validated with standard results using Independent Monte Carlo simulation. The influence of scattering in the material properties has been examined by allowing the coefficients of variation \((SD/mean)\) changing from 0 to 20\% [16]. The schematics procedure of DISFEM is given in Fig. 2. Validation study for random hygrothermal non-linear bending of material properties \((E_{22})\), plate thickness ratio \((a/h)=30\), angle ply \([\pm45^\circ]_T\) square laminated plate, rise in temperature \((\Delta T)=200^\circ C\), rise in moisture percentage \((\Delta C=2\%)\), simple support SSSS S2, fiber volume fraction \((V_f) =0.6\), load deflection \((Q)=100\) and/or random hygrothermal non-linear bending of geometric properties \((h)\), plate thickness ratio \((a/h)=40\), angle ply \([\pm45^\circ]_T\) square laminated plate, rise in temperature \((\Delta T)=200^\circ C\), rise in moisture in percentage \((\Delta C=2\%)\), simple support SSSS S2, fiber volume fraction \((V_f) =0.6\), load deflection \((Q)=100\) is presented. It is seen that present FOPF results are satisfactory with the MCS results as shown in Fig 4(a)&(b). For the MCS approach, the sample values are generated using MATLAB to fit it desired mean and Standard Deviation assuming...
Gaussian probabilistic distribution function (PDF). However, the DISFEM does not put any limitation as regards to PDF of the material properties which is an advantageous over the MCS. The convergence of MCS results is studied by taking different number of samples which are given input to the present deterministic Eq. (16) and the response is calculated numerically to obtain the statistics of the sample of response.

From the convergence study, it is experience that 12000 samples are sufficient to give desired statistics for the present problem. It is observed the present results are very close to MCS results. This indicates the good accuracy of the present formulation for the range of COV considered.

4.3 Sensitiveness of the second order deflection response to input parameters

Table 2 shows the effects of load deflection (Q) and variation of individual random system property bi, $[(i =1 to 9), (7..9),(10) and (11)] = 0.10$ on the dimensionless mean($W_{0nl}$) and coefficient of variation ($W_{nl}$) of hygrothermal transverse central deflection of angle ply [$\pm 45\degree$] square laminated composite plates subjected to uniform constant temperature and moisture (U.T), in-plane bi-axial compression, $a/h=20$, with simple support S2 boundary conditions. The dimensionless nonlinear mean hygrothermal transverse central deflections ($W_{0l}$) are given in brackets. Load deflection $(Q) = 100$, fiber volume fraction $(V_f) =0.6$. It is observed that without considering moisture and temperature for plates the COV of hygrothermal central deflection for $V_{12}$ is significant. On increasing moisture and temperature concentration the mean hygrothermal transverse central deflection decreases.

The effects of area ratios $(a/h)$, load deflection $(Q)$ and random input variables $b_i$, $[(i =1 to 9), (7..9),(10) and (11)] = 0.10$ on the dimensionless non-linear mean $(W_{0nl})$ and coefficient of variation $(W_{nl})$ of hygrothermal transverse central deflection of angle ply [$\pm 45\degree$] square laminated composite plate subjected in-plane bi-axial compression with simple support S2 boundary conditions. It is seen that on increase of plate thickness ratio the expected mean central deflection decreases, however for higher load deflection it is in increasing order. On increasing moisture and temperature the mean transverse central deflection further decreases. The COV of hygrothermal transverse central deflection in both cases significantly varies as shown in Table 3.

Table 4 shows the effects of aspect ratios $(a/b)$, load deflection $(Q)$ and random input variables $b_i$, $[(i =1 to 9), (7..9),(10) and (11)] = 0.10$ on the dimensionless non-linear mean $(W_{0nl})$ and coefficient of variation $(W_{nl})$ of hygrothermal transverse central deflection of angle ply [$\pm 45\degree$] square laminated composite plates subjected to in-plane bi-axial compression with simple support S2 boundary conditions in hygrothermal environments. Plate thickness ratio $(a/h)=40$, fiber volume fraction $(V_f) =0.6$. It is seen that the effects of aspect ratio is significant for the rectangular plates with increasing load deflection, there are further variations for mean and COV of transverse central deflection when the rectangular plates are subjected to rise in moisture and temperatures.

Table 5 shows the effects of support conditions, load deflection $(Q)$ and random input variables $b_i$, $[(i =1 to 9), (7..9),(10) and (11)] = 0.10$ on the dimensionless non-linear mean $(W_{0nl})$ and coefficient of variation $(W_{nl})$ of hygrothermal transverse central deflection, plate thickness ratio $(a/h)=50$, of angle ply [$\pm 45\degree$] square laminated composite plates subjected to in-plane bi-axial compression and fiber volume fraction $(V_f) =0.6$. It is noticed that simple supported (S2) and CSCS supported plates are significantly influenced by variation of load deflection, rise in moisture and temperature. The effects for combined random input variables are significant.

The effects of Lay-up, load deflection $(Q)$ with random input variables $b_i$, $[(i =1 to 9), (7..9),(10) and (11)] = 0.10$ on the dimensionless non-linear mean $(W_{0nl})$ and coefficient of variation $(W_{nl})$ of hygrothermal transverse central deflection, plate thickness ratio $(a/h)=50$, of laminated square composite plate, simple support S2 boundary conditions subjected to in-plane bi-axial compression and fiber volume fraction $(V_f) =0.6$. It is noticed that plates with cross ply and antisymmetric are highly influenced for mean and COV of hygrothermal transverse central deflection on increasing load deflection. However the values are further increased when plates are subjected to rise in moisture and temperatures is shown in Table 6.

Table 7 shows the effects of fiber volume fraction $(V_f)$, load deflection $(Q)$ and random input variables $b_i$, $[(i =1 to 9), (7..9),(10) and (11)] = 0.10$ on the dimensionless non-linear mean $(W_{0nl})$ and coefficient of variation $(W_{nl})$ of
Hygrothermal transverse central deflection of angle ply [±45°]2T laminated composite plates subjected to in-plane bi-axial compression with simple support S2 boundary conditions. Plate thickness ratio (a/h) = 40.

It is observed that effects of volume fraction with increase in load deflection mean and COV hygrothermal transverse central deflection of all combined random input variables is significant when there is rise in moisture and temperatures.

The effects of temperature and moisture rise (ΔT, ΔC), load deflection (Q) and random input variables bi, [(i = 1 to 9), (7..9), (10) and (11)] = 0.10 on the dimensionless non-linear mean (W_{nl}) and coefficient of variation (W_{nl}) of hygrothermal transverse central deflection, plate thickness ratio (a/h) = 20, of angle ply [±45°]2T laminated square composite plate subjected to in-plane bi-axial compression and fiber volume fraction (Vf) = 0.6.

It is noticed that increase in moisture and temperature with load deflection is significant for both mean transverse central deflection and random input variables.

**Conclusions**

A DISFEM probabilistic procedure is presented to study the second order statistics, i.e., mean and COV of transverse central deflection of laminated composite plate with randomness in material properties, coefficients of thermal expansion, coefficients of hygroscopic expansion, geometric parameters and lateral loading. The effects of the mean value of lateral load combination of multiple random variables varying simultaneously or individually, plate geometric parameters, boundary supporting and various modes of temperature and moisture change are addressed in analysis.

The following conclusions are noted from this study.

1. The COV of the transverse central deflection shows different sensitivity to different system properties. The sensitivity changes with the lay-up sequence, the plate to side ratio, the plate aspect ratio, fiber volume fractions, the boundary conditions, temperature and moisture increments, material properties and geometric parameters.

2. Among the different system properties studied, the elastic moduli, lamina plate thickness lateral loading have dominant effect on the COV of the transverse central deflection as compared to other system properties subjected to uniform and uniform temperature and moisture distribution. The strict control of these random parameters is therefore, required if high reliability of the laminated composite is desired.

3. The coefficient of variation of the plate increases as distribution in temperature & moisture and lateral pressure increases. This bring out importance of considering hygrothermal loading along with lateral pressure as one of the essential parameters from design point of view. In general, the rectangular plate is more sensitive as compared to the square plate.

4. The plate with all edges clamped support conditions is more desirable as compared to other support conditions from sensitivity point of view. The effect of randomness in thermal expansion coefficients, moisture expansion coefficients, fiber volume fractions, lamina plate thickness and lateral load on the coefficient of variation of nonlinear transverse central deflection subjected to lateral pressure and hygrothermal loading is quite significant.

**Figure captions**

1. Geometry of laminated composite plate

2. (a) Schematic of stochastic analysis procedure.

(b) Flow chart of solution procedure of stochastic nonlinear bending problem

Schematic of various boundary conditions

4. (a) Validation of present DISFEM results with independent MCS results for only one material property E22 varying subjected lateral loading having SSSS (S2) support condition with uniform temperature and moisture distribution.

(b) Validation of present DISFEM results with independent MCS results for only one geometric property hvarying subjected lateral loading having SSSS (S2) support condition with uniform temperature and moisture distribution.
FIG. 1 GEOMETRY OF LAMINATED COMPOSITE PAATE

Displacement field Model using HSDT

- Linear strain vector
- Nonlinear strain vector in von-Karman sense

Stress strain relationship in a plane
Stress state

- Strain energy of the plate
- Strain energy due to nonlinear elastic foundations
- Potential energy due to hygrothermal stresses
- Work done due to external transverse load

Physical problem with uncertainties

Finite element discretization and assembly

Apply boundary conditions and solve discretized equation using minimum potential energy
Hygrothermoelastic Nonlinear Flexural Response of Graphite Epoxy Laminated Composite Plates

Deterministic analysis

Obtain mean structural response

Stochastic analysis

FIG. 2(a) SCHEMATIC OF STOCHASTIC ANALYSIS PROCEDURE

MCS
(Assign a random sample of material properties with repeated over sampling)

Obtain structural response for mean and standard deviation of nonlinear central deflection

FOPT
(Solution of sensitivity equation)

Start

Evaluate elemental stiffness and elemental force vector \([k], [f]\)

Formulate global stiffness and force vector \([K_s], [F]\)

Formulate generalized (nonlinear bending) eigen value problem \([K_s][W]=[F]\)

Divide global stiffness matrix \([K_s]\), displacement vector and force vector as the sum of its mean and a zero mean random variable where \([K_s]=K_s^0 + [K_s^0] + [K_o] + [K_o^0]\) also \([K_s]=K_s^0 + [K_o] + [W]=[W^0] + [W^r] + [F]=[F^0] + [F^r]\)

Using Taylor’s series expansion and neglecting the second and higher order terms, the divided form of above parameters are substituted into linear bending problem equation and equating the same order of magnitude

Zeroth order deterministic perturbation equation

\([K_s^0][W^0]=[F^0]\)

First order random perturbation equation

\([K_s^0][W^r] + [K_o][W^0]=[F^r]\)

Evaluate the deterministic displacement vector \([W^0]=K_s^0^{-1}[F^0]\)

Evaluate random displacement vector \(\frac{dW^r}{d\theta} - \frac{1}{2} \frac{d^2W^r}{d\theta^2}, 0 < \theta < 1\)

FIG. 2(b) FLOW CHART OF SOLUTION PROCEDURE OF STOCHASTIC NONLINEAR BENDING PROBLEM
FIG. 3 SCHEMATIC OF VARIOUS BOUNDARY CONDITIONS

FIG. 4(a) VALIDATION OF PRESENT DISFEM RESULTS WITH INDEPENDENT MCS RESULTS FOR ONLY ONE MATERIAL PROPERTY $E_{22}$ VARYING SUBJECTED LATERAL LOAD IN HAVING SSSS (S2) SUPPORT CONDITION WITH UNIFORM TEMPERATURE AND MOISTURE DISTRIBUTION.
FIG. 4(b) VALIDATION OF PRESENT DISFEM RESULTS WITH INDEPENDENT MCS RESULTS FOR ONLY ONE GEOMETRIC PROPERTY HAVING SUBJECTED TO LATERAL LOADING HAVING SSSS(S2) SUPPORT CONDITION WITH UNIFORM TEMPERATURE AND MOISTURE DISTRIBUTION.

TABLE 1. HYGROTHERMAL EFFECTS ON THE NONLINEAR BENDING BEHAVIOR OF ANGLE PLY ($\pm45^\circ$) Laminate SQUARE PLATE WITH LOAD DEFORMATIONS ($Q=Q \text{ b/}E_{22}$), FIBER VOLUME FRACTION (VF) = 0.6, PLATE THICKNESS RATIO (A/H) = 10, SIMPLE SUPPORT SSSS(S2) BOUNDARY CONDITIONS.

<table>
<thead>
<tr>
<th>Q</th>
<th>Non-dimensional Hygrothermal Bending Load ($W_{nl}$)</th>
<th>Ref. [26]</th>
<th>Present HSDT</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta T=00, \Delta C=0%$</td>
<td>$\Delta T=300, \Delta C=3%$</td>
<td>$\Delta T=00, \Delta C=0%$</td>
</tr>
<tr>
<td>100</td>
<td>0.6683</td>
<td>0.5957</td>
<td>0.6687</td>
</tr>
<tr>
<td>150</td>
<td>0.8243</td>
<td>0.6717</td>
<td>0.8422</td>
</tr>
<tr>
<td>200</td>
<td>0.9400</td>
<td>0.7161</td>
<td>0.9397</td>
</tr>
</tbody>
</table>

TABLE 2. EFFECTS OF LOAD DEFORMATION ($Q$) AND VARIATION OF INDIVIDUAL RANDOM SYSTEM PROPERTY, $b_i$, ([i=1 TO 11], = 0.10) ON THE DIMENSIONLESS MEAN($W_{nl}$) AND COEFFICIENT OF VARIATION ($W_{nl}$ OF HYGROTHERMAL TRANSVERSE CENTRAL DEFLECTION OF ANGLE PLY ($\pm45^\circ$) SQUARE LAMINATED COMPOSITE PLATES SUBJECTED TO UNIFORM CONSTANT TEMPERATURE AND MOISTURE ($U,T$), IN-PLANE BI-AXIAL COMPRESSION, A/H=20, WITH SIMPLE SUPPORT S2 BOUNDARY CONDITIONS. THE DIMENSIONLESS NONLINEAR MEAN HYGROTHERMAL TRANSVERSE CENTRAL DEFLECTIONS ($W_{nl}$) MEAN HYGROTHERMAL TRANSVERSE CENTRAL DEFLECTIONS ($W_{nl}$) ARE GIVEN IN BRACKETS. LOAD DEFLECTION ($Q$) = 100, FIBER VOLUME FRACTION (VF) = 0.6.

<table>
<thead>
<tr>
<th>$E_{ij}$</th>
<th>(Q)</th>
<th>COV, $W_{nl}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\Delta T=0^\circ C, \Delta C=0.00$</td>
</tr>
<tr>
<td>$E_{11} (i=1)$</td>
<td>100</td>
<td>(0.5031) 2.55e-04</td>
</tr>
<tr>
<td></td>
<td>150</td>
<td>(0.7027) 2.66e-04</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>(0.8760) 2.73e-04</td>
</tr>
<tr>
<td>$E_{22} (i=2)$</td>
<td>100</td>
<td>0.0022 3.52e-05</td>
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<tr>
<td></td>
<td>200</td>
<td>4.90e-04 2.06e-04</td>
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<tr>
<td>$G_{12} (i=3)$</td>
<td>100</td>
<td>0.0031 7.78e-04</td>
</tr>
<tr>
<td></td>
<td>150</td>
<td>0.0032 7.53e-04</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>0.0045 8.73e-04</td>
</tr>
<tr>
<td>a/h</td>
<td>Q</td>
<td>( G_1 (i=4) )</td>
</tr>
<tr>
<td>-----</td>
<td>-----</td>
<td>-----------------</td>
</tr>
<tr>
<td></td>
<td></td>
<td>100</td>
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<tr>
<td></td>
<td></td>
<td>2.39e-05 3.20e-05 4.15e-05 5.36e-05</td>
</tr>
</tbody>
</table>

TABLE 3 : THE EFFECTS OF THICKNESS RATIOS (a/h), LOAD DEFLECTION (Q) AND RANDOM INPUT VARIABLES \( B_i \), \([i=1 to 9], (7,9),(10) \) AND (11) = 0.10 ON THE DIMENSIONLESS NON-LINEAR MEAN (\( W_{avl} \)) AND COEFFICIENT OF VARIATION (\( W_{avl} \)) OF HYDROTERMAL TRANSVERSE CENTRAL DEFLATION OF ANGLE PLY\([\pm 45^\circ]_2 \) SQUARE LAMINATED COMPOSITE PLATE SUBJECTED TO IN-PLANE BI-AXIAL COMPRESSION WITH SIMPLE SUPPORT S2 BOUNDARY CONDITIONS AND VOLUME FRACTION (VF) = 0.6.
### TABLE 4

<table>
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<th>a/b</th>
<th>Q</th>
<th>(TID)</th>
<th>(TID)</th>
</tr>
</thead>
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<tr>
<td></td>
<td></td>
<td>(ΔT = 0°C, ΔC = 0.0)</td>
<td>(ΔT = 100°C, ΔC = 0.01)</td>
</tr>
<tr>
<td></td>
<td>mean</td>
<td>COV, W̄_bi</td>
<td>mean</td>
</tr>
<tr>
<td></td>
<td>(i=1..9)</td>
<td>(i=7..9)</td>
<td>(i=10)</td>
</tr>
<tr>
<td>1.0</td>
<td>100</td>
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<td>0.0164</td>
</tr>
<tr>
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<td>0.5684</td>
<td>0.0143</td>
</tr>
<tr>
<td></td>
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### TABLE 5

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<th>(TID)</th>
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</tr>
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<td>(ΔT = 0°C, ΔC = 0.0)</td>
<td>(ΔT = 100°C, ΔC = 0.01)</td>
</tr>
<tr>
<td></td>
<td>mean</td>
<td>COV, W̄_bi</td>
<td>mean</td>
</tr>
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TABLE 6  EFFECTS OF LAY-UP, LOAD DEFLECTION (Q) WITH RANDOM INPUT VARIABLES BI, [[/1 = 1 TO 9], (7..9),(10) AND (11)] = 0.10 ON THE DIMENSIONLESS NON-LINEARMEAN(WNL) AND COEFFICIENT OF VARIATION (WNL) OF HYDROTHERMAL TRANSVERSE CENTRAL DEFLECTION, PLATE THICKNESS RATIO (A/H) ~50, OF LAMINATED SQUARE COMPOSITE PLATE, SIMPLE SUPPORT S2 BOUNDARY CONDITIONS SUBJECTED TO IN-PLANE BI-AXIAL COMPRESSION AND FIBER VOLUME FRACTION (VF) = 0.6.

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TABLE 7  EFFECTS OF FIBER VOLUME FRACTION (VF), LOAD DEFLECTION (Q) WITH RANDOM INPUT VARIABLES BI, [[/1 = 1 TO 9], (7..9),(10) AND (11)] = 0.10 ON THE DIMENSIONLESS NON-LINEARMEAN (WNL) AND COEFFICIENT OF VARIATION (WNL) OF HYDROTHERMAL TRANSVERSE CENTRAL DEFLECTION OF ANGLE PLY [±45°]2T LAMINATED COMPOSITE PLATES SUBJECTED TO IN-PLANE BI-AXIAL COMPRESSION WITH SIMPLE SUPPORT S2 BOUNDARY CONDITIONS, PLATE THICKNESS RATIO (A/H) = 40.

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REFERENCES


Nigam NC, Narayanan S. Applications of random vibrations, Narosa, New Delhi, 1994.


APPENDIX

\[ (A_y, B_y, D_y, E_y, F_y, H_y) = \int_{A_1}^{A_2} q_y (1, z, z^2, z^3, z^4) dz; \ (i,j) = 1,2,6 \]

\[ (A_y, D_y, F_y) = \int_{A_1}^{A_2} q_y (1, z^2, z^4) dz; \ (i,j) = 4,5 \]

\[ [D_3] = \begin{bmatrix} [A_1] & [0] \\ [B] & [0] \\ [E] & [0] \end{bmatrix}, [D_5] = [D_3]^T \text{ and } [D_3] = \begin{bmatrix} [A_1] & [0] \\ [0] & [A_1] \end{bmatrix} \]

\[ [K_y] = \sum_{i=1}^{N} \int_{A_1}^{A_2} \begin{bmatrix} B_y^{(i)} \\ D_y^{(i)} \end{bmatrix}^T \begin{bmatrix} B_y^{(i)} \\ D_y^{(i)} \end{bmatrix} dA; \]

\[ [K_y] = \sum_{i=1}^{N} \int_{A_1}^{A_2} \begin{bmatrix} B_y^{(i)} \\ D_y^{(i)} \end{bmatrix}^T [D] \begin{bmatrix} B_y^{(i)} \\ D_y^{(i)} \end{bmatrix} dA \]

\[ [K_{NL}(q)] = \int_{A_1}^{A_2} \begin{bmatrix} B_{NL} \\ D_{NL} \end{bmatrix}^T [D] \begin{bmatrix} B_{NL} \\ D_{NL} \end{bmatrix} dA + \frac{1}{2} \int_{A_1}^{A_2} \begin{bmatrix} B_{NL} \\ D_{NL} \end{bmatrix}^T [D] \begin{bmatrix} B_{NL} \\ D_{NL} \end{bmatrix} dA + \frac{1}{2} \int_{A_1}^{A_2} \begin{bmatrix} B_{NL} \\ D_{NL} \end{bmatrix}^T [D] \begin{bmatrix} B_{NL} \\ D_{NL} \end{bmatrix} dA \]

\[ [K_e] = \int_{A_1}^{A_2} \begin{bmatrix} B_e \\ D_e \end{bmatrix}^T [N] dA = \int_{A_1}^{A_2} [G]^T [N] [G] dA \]

\[ [K_f] = \frac{1}{2} \int_{A_1}^{A_2} \begin{bmatrix} B_f \\ D_f \end{bmatrix}^T [D_f] \begin{bmatrix} B_f \\ D_f \end{bmatrix} dA \]
\[ \{q\} = \sum_{r=1}^{NE} \{A\}^{(r)} \]

\[ [F] = \sum_{i=1}^{d} \int_{c_i} \left( [B_{\nu}^{(r)}]^T [N^{HT}] + [B_{\omega}^{(r)}]^T [M^{HT}] + [B_{\zeta}^{(r)}]^T [P^{HT}] \right) dA \]